Why Horizontal & Vertical Shifts Really Work As They Do

1) Why does *adding* a positive number to x move (or translate) a graph to the *left*? Why does *subtracting* a positive number to x move a graph to the *right*?
2) Why does adding a positive number to a function $f(x)$ move its graph up? Why does subtracting a positive number from a function move its graph down?

The function notation of your college algebra textbook provides a simple and elegant answer.

To answer the first question, consider a function $y = f(x)$

If we add a positive constant h to x, the graph of $f(x)$ will shift over, but in which direction?

$y = f(x+h)$ How does the graph change?

Consider an initial point (x_1,y_1) on the graph such that

$f(x_1) = y_1$

Input a second x-value$(x_1–h)$ into the shifted function $f(x+h)$ to get:

$f(\ (x_1–h) + h) = f(x_1 – h + h) = f(x_1) = y_1$

Recall that the point before the shift was (x_1,y_1).

The point after the shift was $(x_1–h,\ y_1)$.

Note that the y-coordinates before and after are the same. There is no movement in the y-direction. Thus, we observe that the shift caused by the addition of the constant h does not move the graph in the vertical direction.

Only the x-coordinates have changed from before and after the shift. Thus, the shift has moved the graph in the horizontal direction. After the shift, the x-coordinate is $x_1–h$. This x-value is h units to the *left* of x_1. Thus, inserting a positive h into the function $f(x+h)$ moves the x-coordinates of all points to the left.

We conclude that $f(x+h)$ represents a horizontal shift to the left of the graph of $f(x)$. A similar argument shows that $f(x–h)$ represents a horizontal shift to the right of the graph of $f(x)$.

To answer the second question, consider adding a positive constant k to the function $f(x)$ to get: $f(x)+k$. In which direction does the graph shift?

Consider the same initial point (x_1,y_1) on the graph such that

$$y_1 = f(x_1)$$

Input the same x-value x_1 into the shifted function $f(x)+k$ to get:

$$y_2 = f(x_1)+k \quad \to \quad y_2 = y_1+k \quad \text{by substitution.}$$

Recall that the point before the shift was (x_1,y_1).

The point after the shift was (x_1,y_2) for which $y_2 = y_1+k$

Note that the x-coordinates before and after are the same. There is no movement in the x-direction. Thus, we observe that the shift caused by the addition of the constant k does not move the graph in the horizontal direction.

Only the y-coordinates have changed from before and after the shift. Thus, the shift has moved the graph in the vertical direction. After the shift, the y-coordinate is y_2. This y-value is equal to y_1+k and is k units *above* y_1. Thus, the positive k in the function $f(x)+k$ moves the y-coordinates of all points *upward*.

We conclude that $f(x)+k$ represents a vertical shift upward for the graph of $f(x)$. A similar argument shows that $f(x)–k$ represents a vertical shift downward for the graph of $f(x)$.

In a later course we can prove that these shifts are *isometries*. An isometry is a transformation that preserves the shape and size of the graph, that is, the graph is not stretched or shrunk after a vertical or horizontal shift

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Completing Multiple Transformations on the Same Function

Transformations or isometries of functions are commutative. Reflections and translations (left/right or up/down shifts) are transformations that are commutative.

For example, consider this function:

1. $f(x) = (3 - x)^2$

This function is quadratic and is isometric with the basic parabola function:

2. $f(x) = x^2$

In other words, the function in (1) is merely the product of one or more transformations on (2).

To help us see the transformations involved, let's factor (1) in a special way:

3. $f(x) = [-1(x-3)]^2$

In this form, we can that this function is the product of a shift to the right by 3 units and a reflection.

We can obtain the graph of function (1) by first reflecting function (2) around the *y*-axis and then shifting the graph to the right by 3 units.

4. *We should be able to obtain the exact same graph by first shifting the graph of function (2) three units to the right and then second, doing the reflection around the y-axis. The order in which we do the two transformations should not matter.*

However, we do not obtain the same graph this second way! (Try it.) What went wrong?

We can resolve this conundrum and remain confident that transformations are indeed commutative by revising the verbiage on reflections.

A reflection about the y-axis occurs when we multiply *x* by a negative:

5. $f(x) = (-x)^2$

Instead, let's call the transformation in function (5) a **reflection around the line *x* = 0**.

This slight change of wording fixes our problem. In statement (4) above, we obtained the wrong graph because we reflected around the y-axis. Instead, if we shift the line of reflection three units to the right as well as the rest of the graph, our new line of reflection is $x = 3$. Reflecting over this line produces the correct graph.

The problem is that if we use the verbiage *reflect around the y-axis*, we may inadvertently believe the line of reflection never moves along with the graph even after a shift left/right because the y-axis is non-movable.