3.1 Quadratic Functions

- Objectives
  - Recognize characteristics of parabolas
  - Graph parabolas
  - Determine a quadratic function's minimum or maximum value.
  - Solve problems involving a quadratic function's minimum or maximum value.

Quadratic functions

f(x)= $ax^2 + bx + c$ graph to be a parabola. The vertex of the parabolas is at (h,k) and “a” describes the “steepness” and direction of the parabola given

$$f(x) = a(x-h)^2 + k$$

Minimum (or maximum) function value for a quadratic occurs at the vertex.

- If equation is not in standard form, you may have to complete the square to determine the point (h,k). If parabola opens up, f(x) has a min., if it opens down, f(x) has a max.

$$f(x) = 2x^2 - 4x + 3$$

$$f(x) = 2(x^2 - 2x) + 3$$

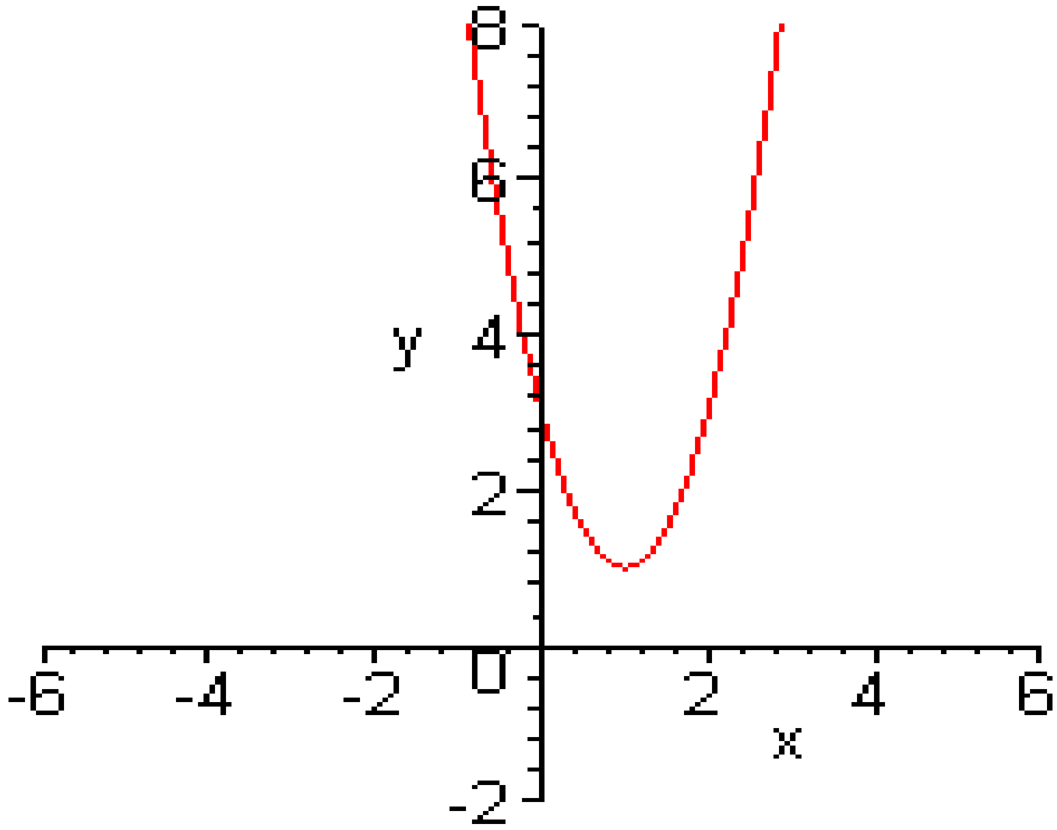
$$f(x) = 2(x-1)^2 - 2 + 3 = 2(x-1)^2 + 1$$

$$(h,k) = (1,1)$$

- This parabola opens up with a “steepness” of 2 and the minimum is at (1,1). (graph on next page)

Graph of

$$f(x) = 2x^2 - 4x + 3 = 2(x-1)^2 + 1$$



3.2 Polynomial Functions & Their Graphs

- Objectives
  - Identify polynomial functions.
  - Recognize characteristics of graphs of polynomials.
  - Determine end behavior.
  - Use factoring to find zeros of polynomials.
  - Identify zeros & their multiplicities.
  - Understand relationship between degree & turning points.
  - Graph polynomial functions.

General form of a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

- The highest degree in the polynomial is the degree of the polynomial.
- The leading coefficient is the coefficient of the highest degreed term.
- Even-degreed polynomials have **both** ends opening up or opening down.
- Odd-degreed polynomials open up on one end and down on the other end.
- WHY? (plug in large values for x and see!!)

Zeros of Polynomials

- When f(x) crosses the x-axis.
- How can you find them?
  – Let f(x)=0 and solve.
  – Graph f(x) and see where it crosses the x-axis.

  What if f(x) just touches the x-axis, doesn't cross it, then turns back up (or down) again?

  This indicates f(x) did not change from pos. or neg. (or vice versa), the zero therefore exists from a square term (or some even power). We say this has a multiplicity of 2 (if squared) or 4 (if raised to the 4^{th} power).

Turning points of a polynomial

- If a polynomial is of degree “n”, then it has at most n-1 turning points.
- Graph changes direction at a turning point.

Graph

$$f(x) = 2x^3 - 6x^2 + 18x$$

$$f(x) = 2x(x^2 - 3x + 9) = 2x(x-3)^2$$

Graph, state zeros & end behavior

$$f(x) = -2x^3 + 12x^2 - 18x = -2x(x^2 - 6x + 9)$$

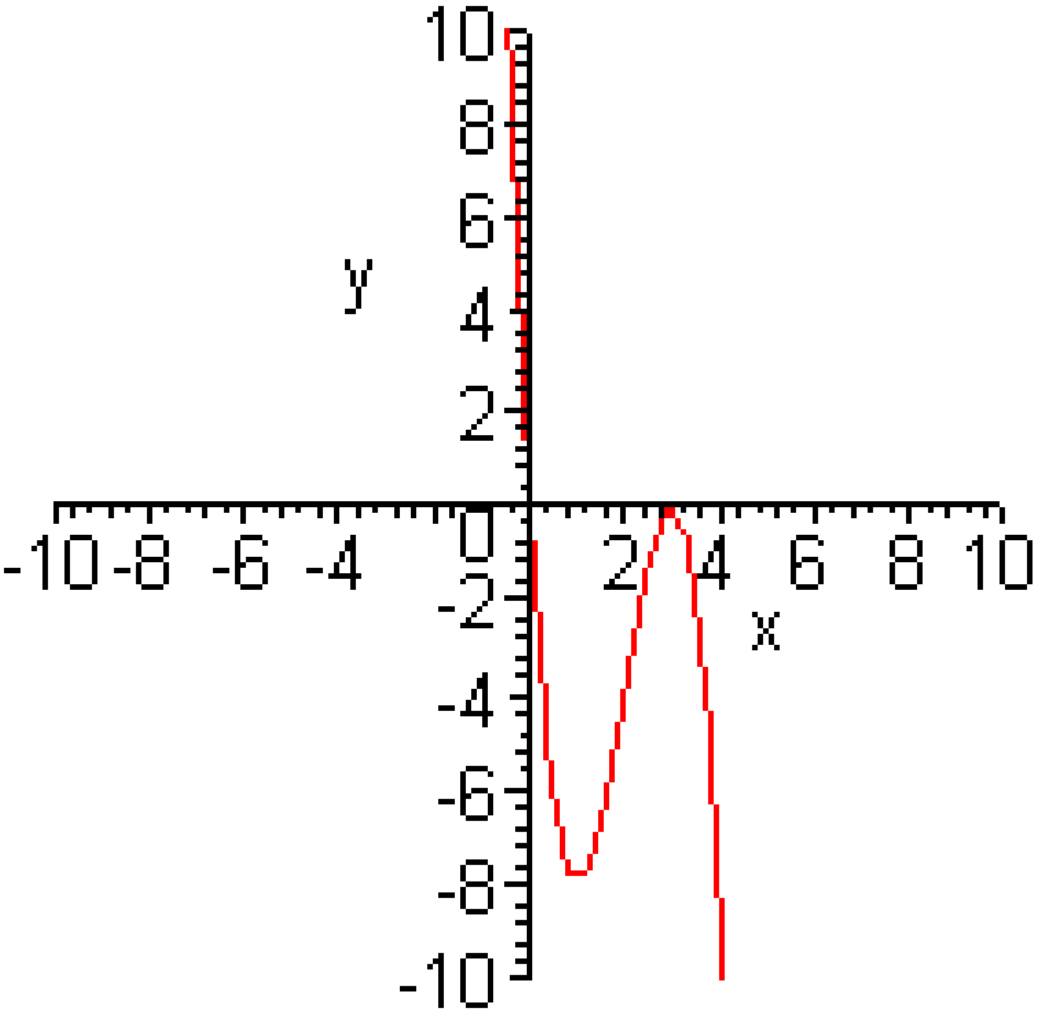
$$f(x) = -2x(x-3)^2$$

- END behavior: 3rd degree equation and the leading coefficient is negative, so if x is a negative number such as -1000, f(x) would be the negative of a negative number, which is positive! (f(x) goes UP as you move to the left.) and if x is a large positive number such as 1000, f(x) would be the negative of a large positive number (f(x) goes DOWN as you move to the right.)
- ZEROS: x = 0, x = 3 of multiplicity 2
- Graph on next page

Graph f(x)

$$f(x) = -2x^3 + 12x^2 - 18x$$

$$f(x) = -2x(x-3)^2$$



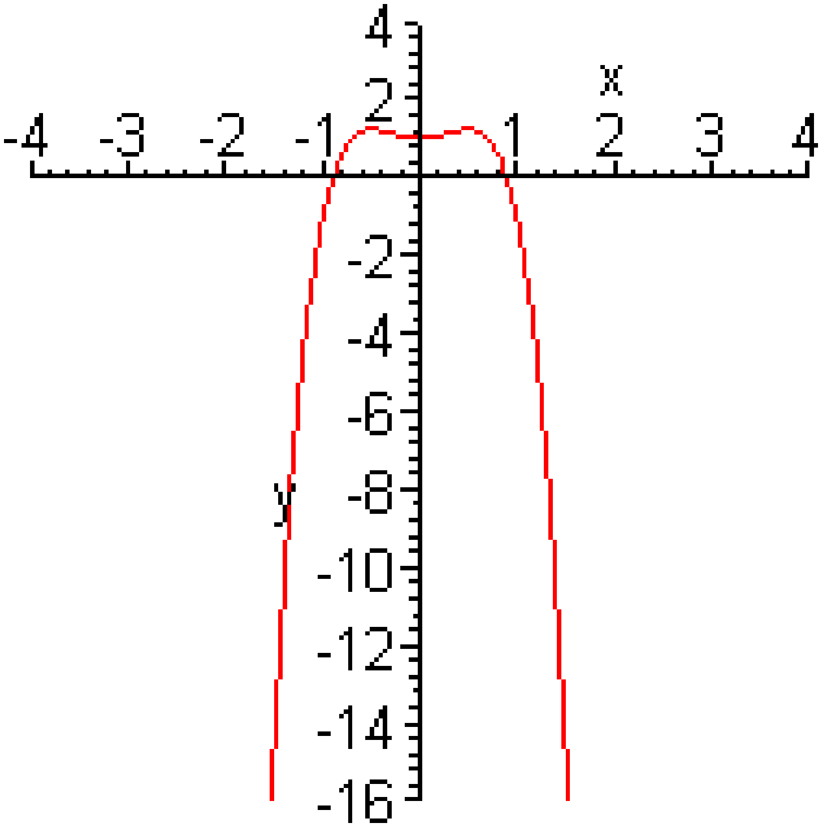
Which function could possibly coincide with this graph?

$1) -7x^5 + 5x + 1$

$2) 9x^5 + 5x^2 - 7x + 1$

$3) 3x^4 + 2x^2 + 1$

$4) -4x^4 + 2x^2 + 1$



Correct Answer: (4)

3.3 Dividing polynomials; Remainder and Factor Theorems

- Objectives
  - Use synthetic division to divide polynomials.
  - Evaluate a polynomials using the Remainder Theorem.
  - Use the Factor Theorem to solve a polynomial equation.

Remainders can be useful!

- The remainder theorem states: If the polynomial $f(x)$ is divided by $(x – c)$, then the remainder is $f(c)$.
- If you can quickly divide, this provides a nice alternative to evaluating $f(c)$.

Factor Theorem

- $f(x)$ is a polynomial, therefore $f(c) = 0$ if and only if $x – c$ is a factor of $f(x)$.
- If we know a factor, we know a zero!
- If we know a zero, we know a factor!

3.4 Zeros of Polynomial Functions

- Objectives
  - Use Rational Zero Thm. to find possible zeros.
  - Find zeros of a polynomial function.
  - Solve polynomial equations.
  - Use the Linear Factorization Theorem to find polynomials, given the zeros.

Rational Root (Zero) Theorem

- If “a” is the leading coefficient and “c” is the constant term of a polynomial, then the only possible rational roots are $\pm$ factors of “c” divided by $\pm$ factors of “a”.
- Example: $f(x) = 6x^5 - 4x^3 - 12x + 4$
- To find the POSSIBLE rational roots of f(x), we need the FACTORS of the leading coefficient and the factors of the constant term. Possible rational roots are $\dfrac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm\left(1, 2, 4, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{2}{3}, \dfrac{4}{3}\right)$

How many zeros does a polynomial with rational coefficients have?

- An nth degree polynomial has a total of n zeros. Some may be rational, irrational or complex.
- For EVEN degree polynomials with RATIONAL coefficients, irrational zeros exist in pairs (both the irrational # and its conjugate).
- If $a+\sqrt{b}$ is a zero, $a-\sqrt{b}$ is a zero
- Complex zeros exist in pairs (both the complex # and its conjugate).
- If $a + bi$ is a zero, $a - bi$ is a zero

3.5 Rational Functions & Their Graphs

- Objectives
  - Find domain of rational functions.
  - Identify vertical asymptotes.
  - Identify horizontal asymptotes.
  - Graph rational functions.
  - Identify slant (oblique) asymptotes.
  - Solve applied problems with rational functions.

Vertical asymptotes

- Look for domain restrictions. If there are values of x which result in a zero denominator, these values would create EITHER a hole in the graph or a vertical asymptote. Which?
- If the factor that creates a zero denominator cancels with a factor in the numerator, there is a hole.
- If you cannot cancel the factor from the denominator, a vertical asymptote exists. Note how the values of $f(x)$ approach positive or negative infinity as the x-values get very close to the value that creates the zero denominator.

Example

- $$f(x) = \frac{3x+7}{x-2}$$

- f(x) is undefined at x = 2
- As $x \to 2^{+}, f(x) \to +\infty$

  $x \to 2^{-}, f(x) \to -\infty$

- Therefore, a vertical asymptote exists at x=2. The graph extends down as you approach 2 from the left, and it extends up as you approach 2 from the right.

What is the end behavior of this rational function?

- If you are interested in the end behavior, you are concerned with very, very large values of x.
- As x gets very, very large, the highest degree term becomes the only term of interest. (The other terms become negligible in comparison.)
- SO, only examine the ratio of the highest degree term in the numerator over the highest degree term of the denominator (ignore all others!)
- As x gets large, $f(x) = \frac{3x+7}{x-2}$ becomes $f(x) = \frac{3x}{x} = 3$
- THEREFORE, a horizontal asymptote exists, y=3

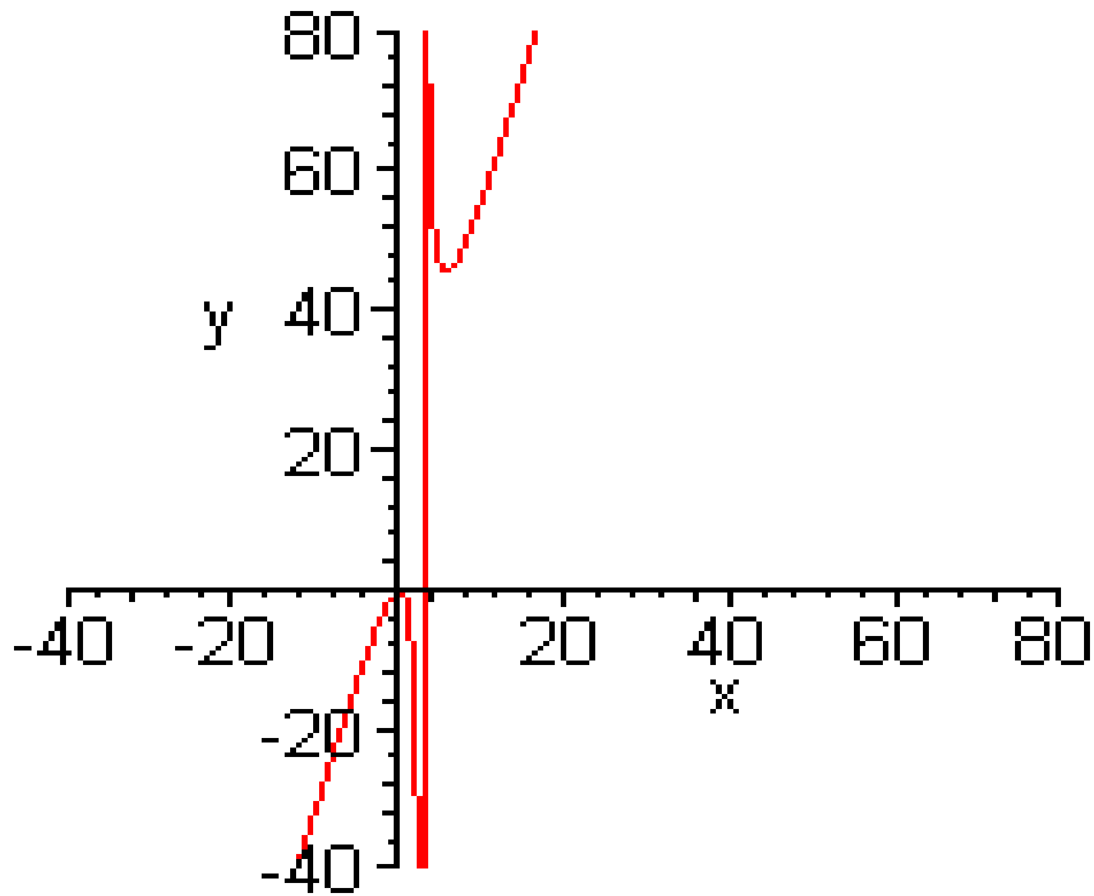
What if end behavior follows a line that is NOT horizontal?

$$f(x) = \frac{8x^2 - 3x + 2}{2x - 2}$$

- The ratio of the highest-degree terms cancels to *4x*
- This indicates we don't have a horizontal asymptote. Rather, the function follows a slanted line with a slope = 4. (becomes *y=4x* as we head towards infinity!)
- To find the exact equation of the slant asymptote, proceed with division, as previously done. The quotient is the slant (oblique) asymptote. For this function, *y = 4x – 11/2*

Graph of this rational function

$$f(x) = \frac{8x^2 - 3x}{2x - 6}$$



What is the equation of the oblique asymptote?

$$f(x) = \frac{4x^2 - 3x + 2}{2x + 1}$$

1. y = 4x – 3
2. y = 2x – 5/2
3. y = 2x – ½
4. y = 4x + 1

Correct Answer: (2)

3.6 Polynomial & Rational Inequalities

- Objectives
  - Solve polynomial inequalities.
  - Solve rational inequalities.
  - Solve problems modeled by polynomial or rational inequalities.

Solving polynomial inequalities

- Always compare the polynomial to zero.
- Factor the polynomial. We are interested in when factors are either pos. or neg., so we must know when the factor equals zero.
- The values of x for which the factors equal zero provide the cut-offs for regions to check if the polynomial is pos. or neg.

$(x - 3)(x + 1)(x - 6) < 0$

- In order for the product of 3 terms to be less than zero (negative), either all 3 terms must be neg. or exactly 1 of them be neg.
- The 3 "cut-off" values are x = 3,-1,6
- The 3 cut-off values create 4 intervals along the x-axis: $(-\infty,-1),(-1,3),(3,6),(6,\infty)$
- Pick a point in each interval & determine if that value for x would make all 3 factors neg. or exactly 1 negative. If so, the function is < 0 on that interval.

$x<-1$, $f(x) < 0$	$-1<x<3$, $f(x) > 0$
$3<x<6$, $f(x)< 0$	$x>6$, $f(x) > 0$

- Solution: $\{x: x < -1 \text{ or } 3 < x < 6\}$

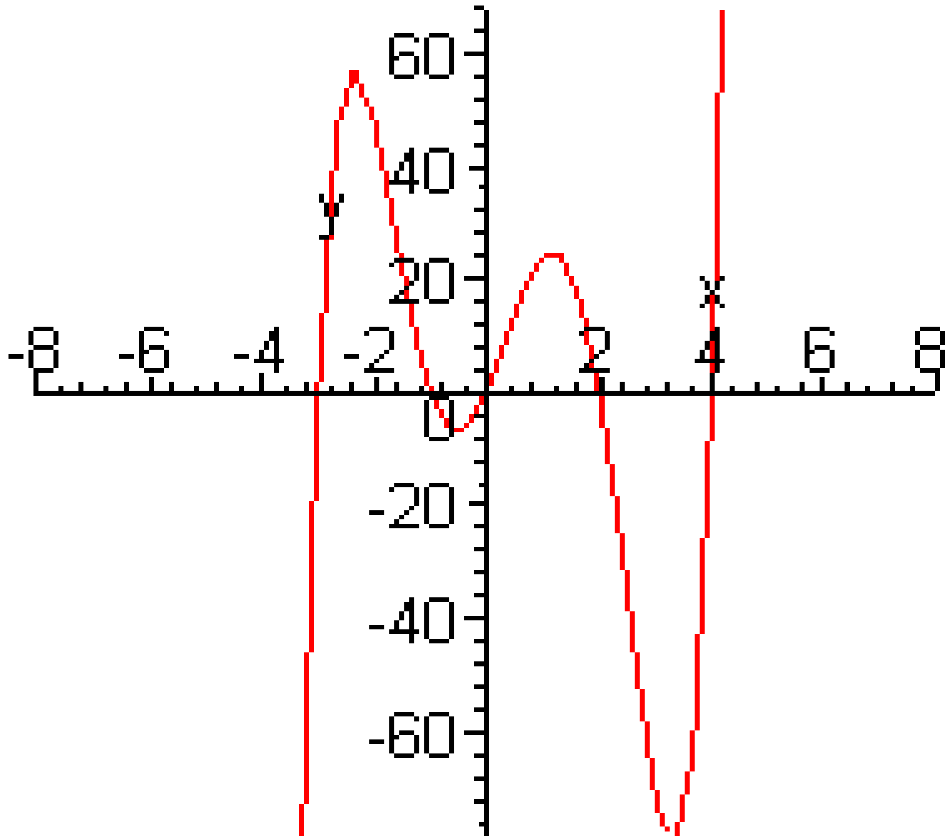
Given the following graph of f(x), give interval notation for x-values such that f(x)>0.

$1)(-3,-1)\cup(0,2)\cup(4,\infty)$

$2)(-\infty,-3)\cup(-1,0)\cup(2,4)$

$3)(-\infty,\infty)$

$4)(-3,4)$



Correct Answer: (1)

Solving rational inequalities

- VERY similar to solving polynomial inequalites EXCEPT if the denominator equals zero, there is a domain restriction. The function COULD change signs on either side of that point.
- Step 1: Compare inequality to zero. (add constant to both sides and use a common denominator to have a rational expression)
- Step 2: Factor both numerator & denominator to find “cut-off” values for regions to check when function becomes positive or negative.