Chapter 4

- Exponential & Logarithmic Functions

4.1 Exponential Functions

- Objectives
  - Evaluate exponential functions.
  - Graph exponential functions.
  - Evaluate functions with base e.
  - Use compound interest formulas.

Definition of exponential function

$$f(x) = b^x$$

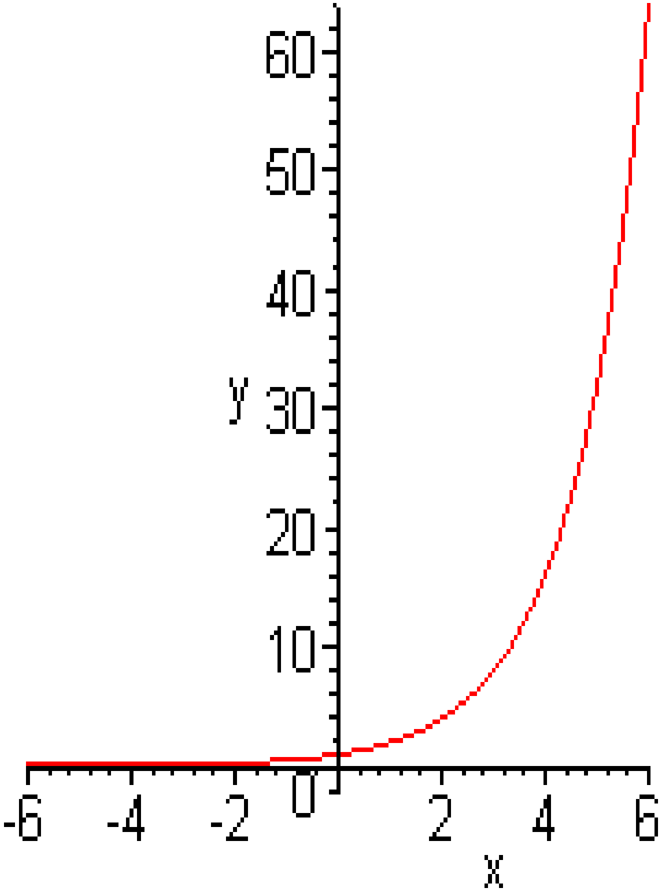
- How is this different from functions that we worked with previously? Some DID have exponents, but NOW, the variable is found in the exponent.
  - (example $f(x) = x^3$ is NOT an exponential function)

Common log

- When the word “log” appears with no base indicated, it is assumed to be base 10.
- Using calculators, “log” button refers to base 10.
- log(1000) means to what EXPONENT do you raise 10 to get 1000?   **3**
- log(10) = -1 (10 raised to the -1 power=1/10)

Graph of an exponential function

- Graph $f(x) = 2^x$
- As x values increase, f(x) grows RAPIDLY
- As x values become negative, with the magnitude getting larger, f(x) gets closer & closer to zero, but with NEVER = 0.
- f(x) NEVER negative



Other characteristics of $f(x) = b^x$

- The y-intercept is the point (0,1) (a non-zero base raised to a zero exponent = 1)
- If the base b lies between 0 & 1, the graph extends UP as you go left of zero, and gets VERY close to zero as you go right.
- Transformations of the exponential function are treated as transformation of polynomials (follow order of operations, x's do the opposite of what you think)

Graph $f(x) = 2^{x+3} - 4$

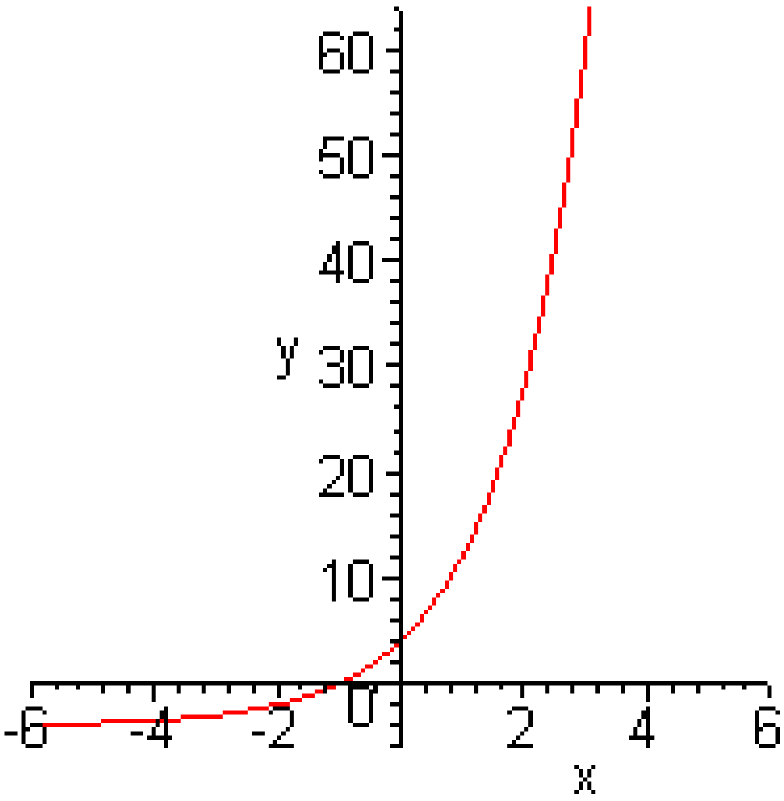
- Subtract 3 from x-values

  (move 3 units left)

- Subtract 4 from y-values

  (move 4 units down)

Note: Point (0,1) has now been moved to (-3,-3)



Applications of exponential functions

- Exponential growth (compound interest!)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- Exponential decay (decomposition of radioactive substances)

4.2 Logarithmic Functions

- Objectives
  - Change from logarithmic to exponential form.
  - Change from exponential to logarithmic form.
  - Evaluate logarithms.
  - Use basic logarithmic properties.
  - Graph logarithmic functions.
  - Find the domain of a logarithmic function.
  - Use common logarithms.
  - Use natural logarithms.

logarithmic and exponential equations can be interchanged

$$f(x) = y = b^x$$

$$x = \log_b y$$

Rewrite the following exponential expression as a logarithmic one.

$$3^{(x+2)} = 7$$

1) $\log_7(x+2) = 3$

2) $\log_3(x+2) = 7$

3) $\log_3(7) = x+2$

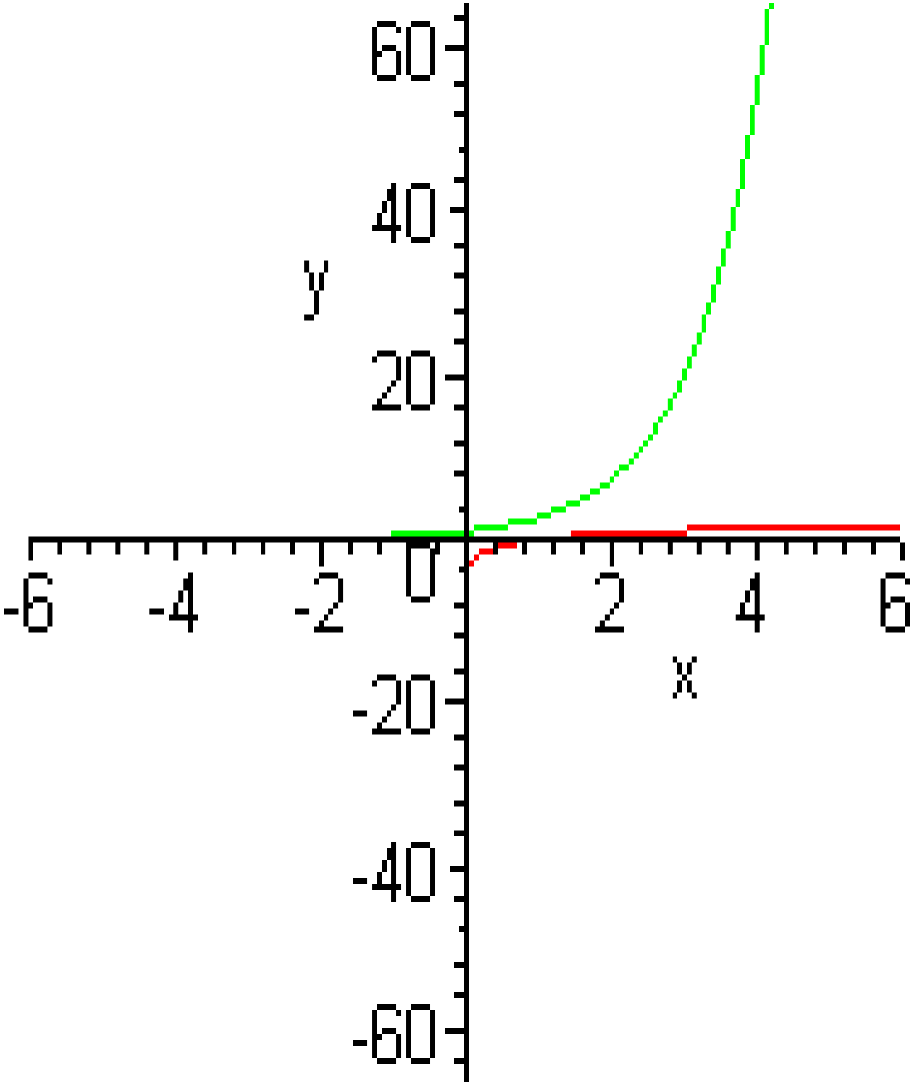
4) $\log_3(x-2) = 7$

Answer: 3

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

- Logarithmic function and exponential function are inverses of each other.
- The domain of the exponential function is all reals, so that's the domain of the logarithmic function.
- The range of the exponential function is x>0, so the range of the logarithmic function is y>0.



Transformation of logarithmic functions is treated as other transformations

- Follow order of operation
- Note: When graphing a logarithmic function, the graph only exists for $x>0$, WHY? If a positive number is raised to an exponent, no matter how large or small, the result will always be POSITIVE!

Domain Restrictions for logarithmic functions

- Since a positive number raised to an exponent (pos. or neg.) always results in a positive value, you can ONLY take the logarithm of a POSITIVE NUMBER.
- Remember, the question is: What POWER can I raise the base to, to get this value?
- DOMAIN RESTRICTION:

$$y = \log_b x, x > 0$$

Common logarithms

- If no value is stated for the base, it is assumed to be base 10.
- log(1000) means, “What power do I raise 10 to, to get 1000?” The answer is 3.
- log(1/10) means, “What power do I raise 10 to, to get 1/10?” The answer is -1.

Natural logarithms

- ln(x) represents the natural log of x, which has a base=e
- What is e? If you plug large values into $\left(1+\frac{1}{x}\right)^x$ you get closer and closer to e.
- logarithmic functions that involve base e are found throughout nature
- Calculators have a button “ln” which represents the natural log. $\log_e x = \ln(x)$

4.3 Properties of logarithms

- Objectives
  - Use the product rule.
  - Use the quotient rule.
  - Use the power rule.
  - Condense logarithmic expressions.

Logarithms are Exponents Rule for logarithms come from rules for exponents

- When multiplying quantities with a common base, we add exponents. When we find the logarithm of a product, we add the logarithms

$$\log_b (M \cdot N) = \log_b M + \log_b N$$

- Example:

$$\log_7 (49x^3) = \log_7 (49) + \log_7 (x^3)$$

$$= \log_7 (7^2) + \log_7 (x^3) = 2 + \log_7 (x^3)$$

Quotient Rule

- When dividing expressions with a common base, we subtract exponents, thus we have the rule for logarithmic functions:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

- Example:

$$\ln\left(\frac{e^x}{15}\right) = \ln(e^x) - \ln(15) = x - \ln(15)$$

Power rule

- When you raise one exponent to another exponent, you multiply exponents.
- Thus, when you have a logarithm that is raised to a power, you multiply the logarithm and the exponent (the exponent becomes a multiplier)

$$\log_b(x^r) = r \cdot \log_b x$$

- Example: Simplify

$$\log\left(1000^x\right) = x \cdot \log(1000) = 3x$$

4.4 Exponential & Logarithmic Equations

- Objectives
  - Use like bases to solve exponential equations.
  - Use logarithms to solve exponential equations.
  - Use the definition of a logarithm to solve logarithmic equations.
  - Use the one-to-one property of logarithms to solve logarithmic equations.
  - Solve applied problems involving exponential & logarithmic equations.

Solving equations

- Use the properties we have learned about exponential & logarithmic expressions to solve equations that have these expressions in them.
- Find values of x that will make the logarithmic or exponential equation true.
- For exponential equations, if the base is the same on both sides of the equation, the exponents must also be the same (equal!)

$$b^M = b^N, M = N$$

Sometimes it is easier to solve a logarithmic equation than an exponential one

- Any exponential equation can be rewritten as a logarithmic one, then you can apply the properties of logarithms
- Example: Solve:

$$5^{2x-1} = 99$$

$$\log_5(99) = 2x - 1$$

$$\frac{\ln(99)}{\ln(5)} = 2x - 1$$

$$2.055 = 2x - 1$$

$$x \approx 1.93$$

SOLVE

$$3^{2x-1} = 5^{x+1}$$

$$\log_3(5^{x+1}) = 2x-1$$

$$(x+1)\cdot\log_3 5 = 2x-1$$

$$(x+1)\cdot\frac{\ln 5}{\ln 3} = 2x-1$$

$$1.46x + 1.46 = 2x-1$$

$$2.46 = .54x$$

$$x \approx 4.56$$

SOLVE

$$\log_4(x+3) - \log_4(5x-1) = 3$$

$$\log_4\left(\frac{x+3}{5x-1}\right) = 3$$

$$4^3 = \frac{x+3}{5x-1} = 64$$

$$\frac{x+3}{5x-1} - \frac{64(5x-1)}{5x-1} = 0$$

$$\frac{x+3-320x+64}{5x-1} = 0$$

$$\frac{-319x+67}{5x-1} = 0$$

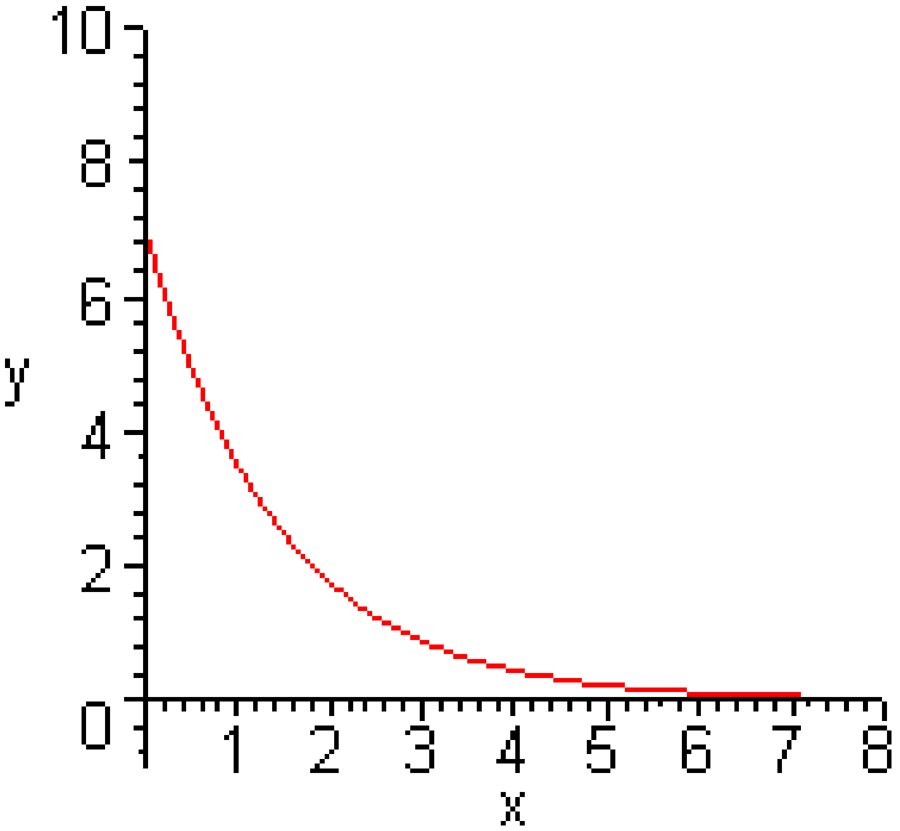
$$x = \frac{67}{319} \approx .21$$

4.5 Exponential Growth & Decay Modeling Data

- Objectives
  - Model exponential growth & decay
  - Model data with exponential & logarithmic functions.
  - Express an exponential model in base e.

Could the following graph model exponential growth or decay?

- 1) Growth model.
- 2) Decay model.



Answer: Decay Model because graph is decreasing.

Exponential Growth & Decay Models

$$A = A_o e^{kt}$$

- A(not) is the amount you start with, t is the time, and k=constant of growth (or decay)
- f k>0, the amount is GROWING (getting larger), as in the money in a savings account that is having interest compounded over time
- If k<0, the amount is SHRINKING (getting smaller), as in the amount of radioactive substance remaining after the substance decays over time