# CHAPTER 6

## MATRICES AND DETERMINANTS

# 6.1 Matrix Solutions to Linear Systems

- Objectives
  - Write the augmented matrix for a linear system
  - Perform matrix row operations
  - Use matrices & Gaussian elimination to solve systems
  - Use matrices & Gauss-Jordan elimination to solve systems

# What is a matrix?

- A set of numbers in rows & columns
- m x n describes the dimensions of the matrix (m rows & n columns)
- The matrix is contained within brackets.
- Example of a 3 x 3 matrix:

$$\begin{bmatrix} 4 & 2 & 5 \\ 1 & -4 & 3 \\ 0 & -3 & 8 \end{bmatrix}$$

# Solving a system of 3 equations with 3 variables

- Graphically, you are attempting to find where 3 **planes** intersect.
- If you find 3 numeric values for (x,y,z), this indicates the 3 planes intersect at that **point**.

# Representing a system of equations in a matrix

- If a linear system of 3 equations involved 3 variables, each column represents the different variables & constant, and each row represents a separate equation.
- Example: Write the following system as a matrix

2x + 3y – 3z = 7

5x + y – 4z = 2

4x + 2y - z = 6

$$\left|\begin{matrix} 2 & 3 & -3 & 7 \\ 5 & 1 & -4 & 2 \\ 4 & 2 & -1 & 6 \end{matrix}\right|$$

# Solving linear systems using Gaussian elimination

- Use techniques learned previously to solve equations (addition & substitution) to solve the system
- Variables are eliminated, but each column represents a different variable
- Perform addition &/or multiplication to simplify rows.  Have one row contain 2 zeros, a one, and a constant.  This allows you to solve for one variable.
- Work up the matrix and solve for the remaining variables.

# A matrix should look like this after Gaussian elimination is applied

$$\begin{vmatrix} 1 & y_1 & z_1 & k_1 \\ 0 & 1 & z_2 & k_2 \\ 0 & 0 & 1 & k_3 \\ & & & \end{vmatrix}$$

# Examples 3 & 4 (p. 541-543)

- These examples outline a stepwise approach to use Gaussian elimination.
- One row (often the bottom row) will contain (0  0  0  1  constant) using this method.
- After the last variable is solved in the bottom equation, substitute in for that variable in the remaining equations.

# Question: Must the last row contain (0 0 0 1 constant)?

- Yes, in Gaussian elimination, but there are other options.
- Look again at example 3, page 539.
- If you multiply the first row by (-1) and add it to the 2nd row, the result is (-2 0 0 -12)
- What does this mean? -2x = -12, x=6
- By making an informed decision as to what variables to eliminate, we solved for a variable much more quickly!
- Next, multiply row 1 by (-1) & add to 3rd row: (-2 2 0 6). Recall x=-6, therefore this becomes
  - -2(6)+2y=6, y=3
  - Now knowing x & y, solve for z in any row (row 2?)
  - 6 + 3 + 2z = 19, z=5 Solution: (6,3,5)

# If this method is quicker, why would we use Gaussian elimination?

- Using a matrix to solve a system by Gaussian elimination provides a standard, **programmable** approach.
- When computer programs (may be contained in calculators) solve systems. This is the method utilized!

# 6.2 Inconsistent & Dependent Systems & Their Applications

- Objectives
  - Apply Gaussian elimination to systems without unique solutions
  - Apply Gaussian elimination to systems with more variables than equations
  - Solve problems involving systems without unique solutions.

# How would you know, with Gaussian elimination, that there are no solutions to your system?

- When reducing your matrix (attempting to have rows contain only 0's, 1's & the constant) a row becomes 0 0 0 0 k
- What does that mean? Can you have 0 times anything equal to a non-zero constant? NO! No solution!
- **Inconsistent system – no solution**

# Graphically, what is happening with an inconsistent system?

- Recall, with 3 variables, the equation represents a plane, therefore we are considering the intersection of 3 planes.
- If a system is inconsistent, 2 or more of the planes may be parallel OR 2 planes could intersect forming 1 line and a different pair of planes intersect at a different line, therefore there is nothing in common to all three planes.

# Could there be more than one solution?

- Yes! If the planes intersect to form a line, rather than a point, there would be **infinitely many** solutions. All pts. lying on the line would be solutions.
- **You can't state infinitely many points, so you state the general form of all points on the line, in terms of one of the variables.**

# What if your system has 3 variables but only 2 equations?

- Graphically, this is the intersection of 2 planes.
- 2 planes cannot intersect in 1 point, rather they intersect in 1 line. (or are parallel, thus no solution)
- The solution is all points on that line.
- The ordered triple is represented as one of the variables (usually z) and the other 2 as functions of that variable: ex: $(z+2, 3z, z)$

# Dependent system

- Notice when there were infinitely many solutions, two variables were stated in terms of the 3rd. In other words, the x & y values are dependent on the value selected for z.
- If there are infinitely many solutions, the system is considered to be **dependent**.

# 6.5 Determinants & Cramer's Rule

- Objectives
  - Evaluate a $2^{nd}$-order determinant
  - Solve a system of linear equations in 2 variables using Cramer's rule
  - Evaluate a $3^{rd}$-order determinant
  - Solve a system of linear equations in 3 variables using Cramer's rule
  - Use determinants to identify inconsistent & dependent systems
  - Evaluate higher-order determinants

# Determinant of a 2x2 matrix

- If A is a matrix, the determinant is $\left|A\right|$

$$A=\begin{bmatrix}3 & -2\\ 4 & 5\end{bmatrix}, \left|A\right|=\begin{vmatrix}3 & -2\\ 4 & 5\end{vmatrix}=3(5)-(-2)4=23$$

# When are determinants useful?

- They can be used to solve a system of equations
- Cramer's Rule

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \ y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

# Finding a determinant of a 3x3 matrix

- More complicated, but it can be done!

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

- It’s often easier to pick your “home row/column” (the one with the multipliers) to be a row/column that has one or more zeros in it.

# Determinants can be used to solve a linear system in 3 variables CRAMER'S RULE

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

# What is D and $D_x, D_y, D_z$

- D is the determinant that results from the coefficients of all variables.
- $D_x$ is the determinant that results when each x coefficient is replaced with the given constants.
- $D_y$ is the determinant that results when each y coefficient is replaced with the given constants.
- $D_z$ is the determinant that results when the z coefficients are replaced with the given constants.

# Find z, given

$$2x + y = 7$$

$$-x + 3y + z = 5$$

$$3x + 2y - 4z = 10$$

1) $$\frac{\begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 7 & 5 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 3 & 2 & -4 \end{vmatrix}}$$

(2) $$\frac{\begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 3 & 2 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 7 \\ -1 & 3 & 5 \\ 3 & 2 & 10 \end{vmatrix}}$$

3) $$\frac{\begin{vmatrix} 2 & 1 & 7 \\ -1 & 3 & 5 \\ 3 & 2 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 3 & 2 & -4 \end{vmatrix}}$$

(4) $$\frac{\begin{vmatrix} 2 & 1 & 7 \\ -1 & 3 & 5 \\ 3 & 2 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 7 \\ -1 & 3 & 1 \\ 3 & 2 & -4 \end{vmatrix}}$$