CHAPTER 8

Sequences, Induction, & Probability

8.1 Sequences & Summation Notation

- Objectives
  - Find particular terms of sequence from the general term
  - Use recursion formulas
  - Use factorial notation
  - Use summation notation

What is a sequence?

- An infinite sequence is a function whose domain is the set of positive integers. The function values, terms, of the sequences are represented by $a_1, a_2, a_3, \ldots a_n \ldots$

- Sequences whose domains are the first n integers, not ALL positive integers, are finite sequences.

Recursive Sequences

- A specific term is given.
- Other terms are determined based on the value of the previous term(s)
- Example: $a_3 = 10, a_{n+1} = 3 \cdot a_n + 1, \mathit{find} : a_1, a_2, a_4$
- $$a_3 = 10 = 3a_2 + 1$$
  $$a_2 = \frac{10 - 1}{3} = 3 = 3a_1 + 1$$
  $$a_1 = \frac{3 - 1}{3} = \frac{2}{3}$$
  $$a_4 = 3a_3 + 1 = 3(10) + 1 = 31$$

Find the 1st 3 terms of the sequence:

$$a_n = \frac{n+1}{n!}$$

- 1)4, 5/2, 6
- 2)4, 5/2, 1
- 3)1, 2, 3
- 4)4, 5, 6

Summation Notation

- The sum of the first n terms, as i goes from 1 to n is given as: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$

- Example:

$$\sum_{i=4}^{8} 5i - 2 = [(5\cdot 4)-2]+[(5\cdot 5)-2]+[(5\cdot 6)-2]+[(5\cdot 7)-2]+[(5\cdot 8)-2]$$

$$= 18 + 23 + 28 + 33 = 102$$

8.2 Arithmetic Sequences

- Objectives
  - Find the common difference for an arithmetic sequence
  - Write terms of an arithmetic sequence
  - Use the formula for the general term of an arithmetic sequence
  - Use the formula for the sum of the first n terms of an arithmetic sequence

What is an arithmetic sequence?

- A sequence where there is a common difference between every 2 terms.
- Example: 5,8,11,14,17,…..
- The common difference (d) is 3
- If a specific term is known and the difference is known, you can determine the value of any term in the sequence
- For the previous example, find the 20th term

Example continued

- The first term is 5 and d=3
- Notice between the 1st & 2nd terms there is 1 (3). Between the 1st & 4th terms there are 3 (3's). Between the 1st & nth terms there would be (n-1) 3's
- 20th term would be the 1st term + 19(3's)

$$a_{20} = 5 + 19(3) = 62$$

The sum of the 1^{st} n terms of an arithmetic sequence

- Since every term is increasing by a constant (d), the sequence, if plotted on a graph (x=the indicated term, y=the value of that term), would be a line with slope= d
- The average of the 1^{st} & last terms would be greater than the 1^{st} term by k and less than the last term by k. The same is true for the 2^{nd} term & the 2^{nd} to last term, etc
- Therefore, you can find the sum by replacing each term by the average of the 1^{st} & last terms (continue next slide)

Sum of an arithmetic sequence

- If there are n terms in the arithmetic sequence and you replace all of them with the average of the 1st & last, the result is:

$$S_n = n \cdot \left(\frac{a_1 + a_n}{2} \right)$$

Find the sum of the 1st 30 terms of the arithmetic sequence if

$$a_1 = -6, d = 6$$

- 1) 81
- 2) 3430
- 3) 2430
- 4) 168

» Answer: sum = 2430

8.3 Geometric Sequences & Series

- Objectives
  - Find the common ratio of a geometric sequence
  - Write terms of a geometric sequence
  - Use the formula for the general term of a geometric sequence
  - Use the formula for the sum of the 1^{st} n terms of a geometric sequence
  - Find the value of an annuity
  - Use the formula for the sum of an infinite geometric series

What is a geometric sequence?

- A sequence of terms that have a common multiplier (r) between all terms
- The multiplier is the ratio between the (n+1)th term & the nth term
- Example: -2,4,-8,16,-32,…
- The ratio between any 2 terms is (-2) which is the value you multiply any term by to find the next term

Given a term in a geometric sequence, find a specified other term

- Example: If 1^{st} term=3 and r=4, find the 14^{th} term
- Notice to find the 2^{nd} term, you multiply 3(4)
- To find the 3^{rd} term, you would multiply 3(4)(4)
- To find the 4^{th} term, multiply 3(4)(4)(4)
- To find the nth term, multiply: 3(4)(4)(4)… (n-1 times)
- 14^{th} term = $3(4)^{13} = 201{,}326{,}592$

- (in a geometric sequence, terms get large quickly!)

Sum of the 1st n terms of a geometric sequence

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

What if 0<r<1 or -1<r<0?

- Examine an example:
- If 1st term=6 and r=-1/3

$$6,-2,\frac{2}{3},\frac{-2}{9},\frac{2}{27},\frac{-2}{81},\frac{2}{243},\ldots$$

- Even though the terms are alternating between pos. & neg., their magnitude is getting smaller & smaller
- Imagine infinitely many of these terms: the terms become infinitely small

Find the Sum of an Infinite Geometric Series

- If -1<r<1 and r not equal zero, then we CAN find the sum, even with infinitely many terms (remember, after a while the terms become infinitely small, thus we can find the sum!)
- If $S_n = \left(\frac{a_1(1-r^n)}{1-r} \right)$ and n is getting very large, then r raised to the n, recall, is getting very, very small…so small it approaches zero, which allows us to replace r raised to the nth power with a zero!
- This leads to: $S_n = \frac{a_1}{1-r}$

Repeating decimals can be considered as infinite sums

- Example: Write .34444444…as an infinite sum
- Separate the .3 from the rest of the number:
- .3444….. = .3 + .044444…..
- .044444….. = .04 + .004 + .0004 + .0004+…
- This is an infinite sum with 1st term=.04,r=.1

$$S_\infty = \frac{.04}{1-.1} = \frac{.04}{.9} = \frac{4}{90} = \frac{2}{45}$$

- .3444….= $\frac{3}{10} + \frac{2}{45} = \frac{27}{90} + \frac{4}{90} = \frac{31}{90}$