Chapter 2 – Section 2

Graphing Quadratic Functions

Let a, b, and c be real numbers $a \neq 0$. The function

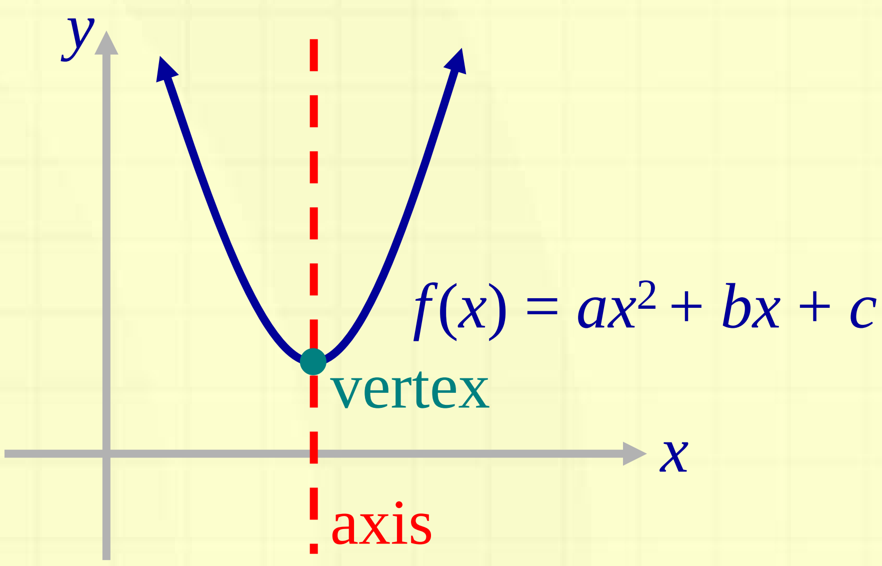
$$f(x) = ax^2 + bx + c$$

is called a **quadratic function**.

The graph of a quadratic function is a **parabola**.

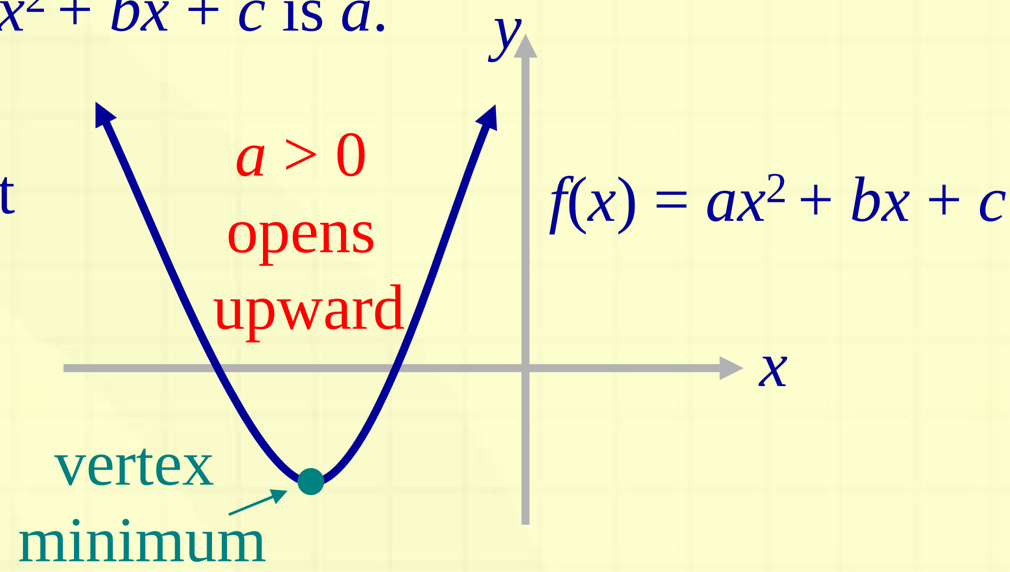
Every parabola is symmetrical about a line called the **axis** (**of symmetry**).

The intersection point of the parabola and the axis is called the **vertex** of the parabola.

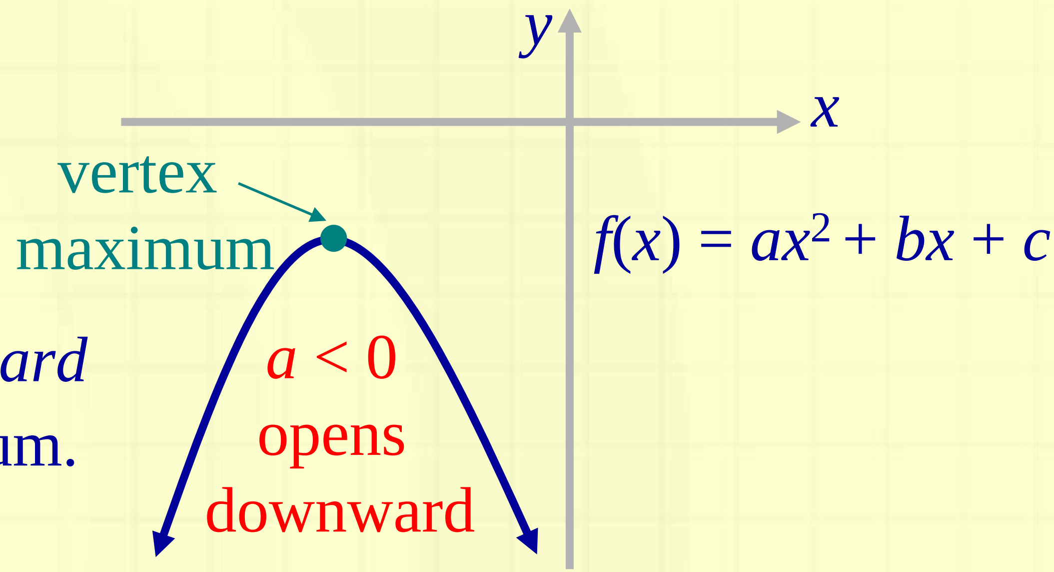


The *leading coefficient* of $ax^2 + bx + c$ is a.

When the leading coefficient is positive, the parabola opens *upward* and the vertex is a *minimum*.



When the leading coefficient is negative, the parabola opens *downward* and the vertex is a maximum.



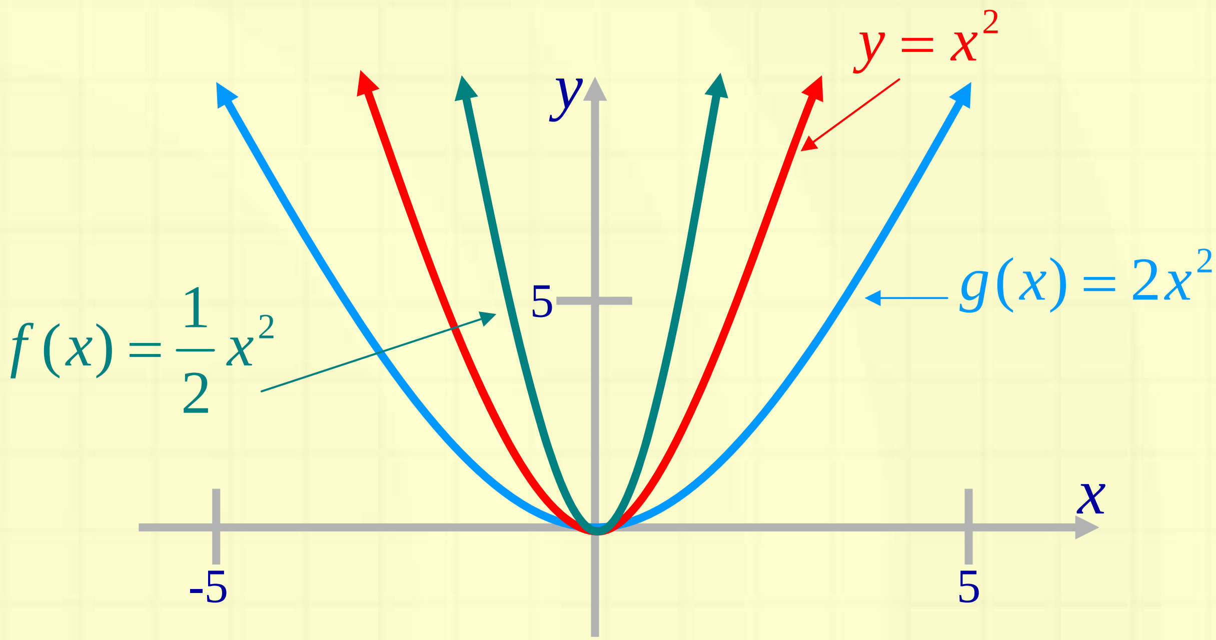
The simplest quadratic functions are of the form

$$f(x) = ax^2 \quad (a \neq 0)$$

These are most easily graphed by comparing them with the graph of $y = x^2$.

Example: Compare the graphs of

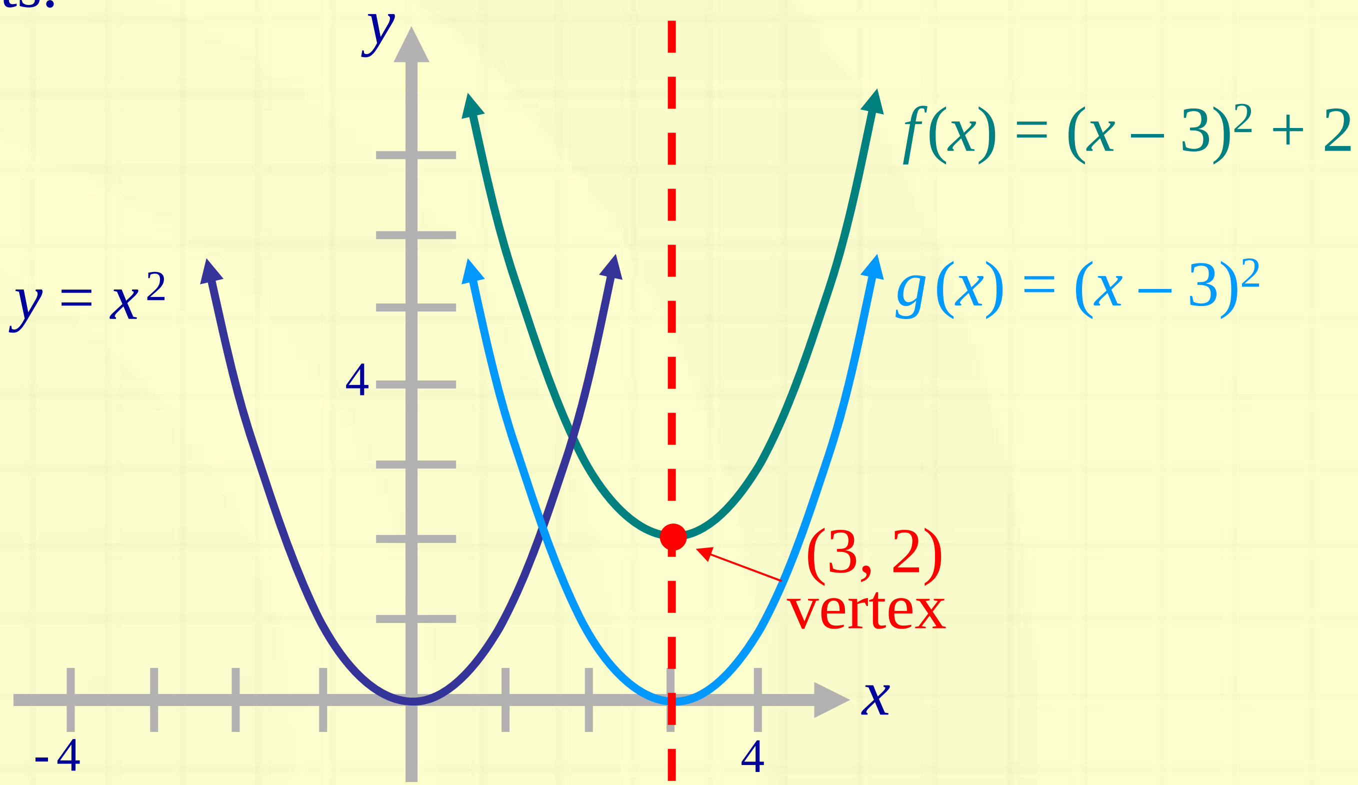
$$y = x^2, \; f(x) = \frac{1}{2}x^2 \text{ and } g(x) = 2x^2$$



Example: Graph $f(x) = (x - 3)^2 + 2$ and find the vertex and axis.

$f(x) = (x - 3)^2 + 2$ is the same shape as the graph of $g(x) = (x - 3)^2$ shifted upwards two units.

$g(x) = (x - 3)^2$ is the same shape as $y = x^2$ shifted to the right three units.



The **standard form** for the equation of a quadratic function is:

$$f(x) = a(x - h)^2 + k \ (a \neq 0)$$

The graph is a parabola opening *upward* if $a > 0$ and opening *downward* if $a < 0$. The *axis* is $x = h$, and the *vertex* is (h, k).

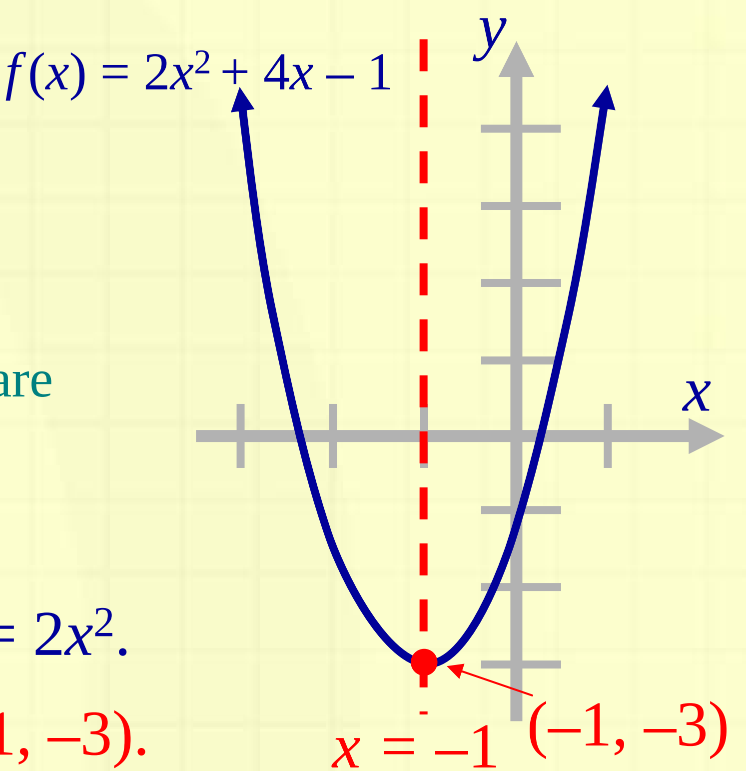
Example: Graph the parabola $f(x) = 2x^2 + 4x - 1$ and find the axis and vertex.

$f(x) = 2x^2 + 4x - 1$ original equation

$f(x) = 2(x^2 + 2x) - 1$ factor out 2

$f(x) = 2(x^2 + 2x + 1) - 1 - 2$ complete the square

$f(x) = 2(x + 1)^2 - 3$ standard form



a > 0 $\rightarrow$ parabola opens upward like $y = 2x^2$.

$h = -1$, $k = -3$ $\rightarrow$ axis $x = -1$, vertex $(-1, -3)$.

Example: Graph and find the vertex and x-intercepts of $f(x) = -x^2 + 6x + 7$.

$f(x) = -x^2 + 6x + 7$	original equation
$f(x) = -(x^2 - 6x) + 7$	factor out -1
$f(x) = -(x^2 - 6x + 9) + 7 + 9$	complete the square
$f(x) = -(x - 3)^2 + 16$	standard form

$a < 0 \rightarrow$ parabola opens downward.

$h = 3,\ k = 16 \rightarrow$ axis $x = 3$, vertex $(3, 16)$.

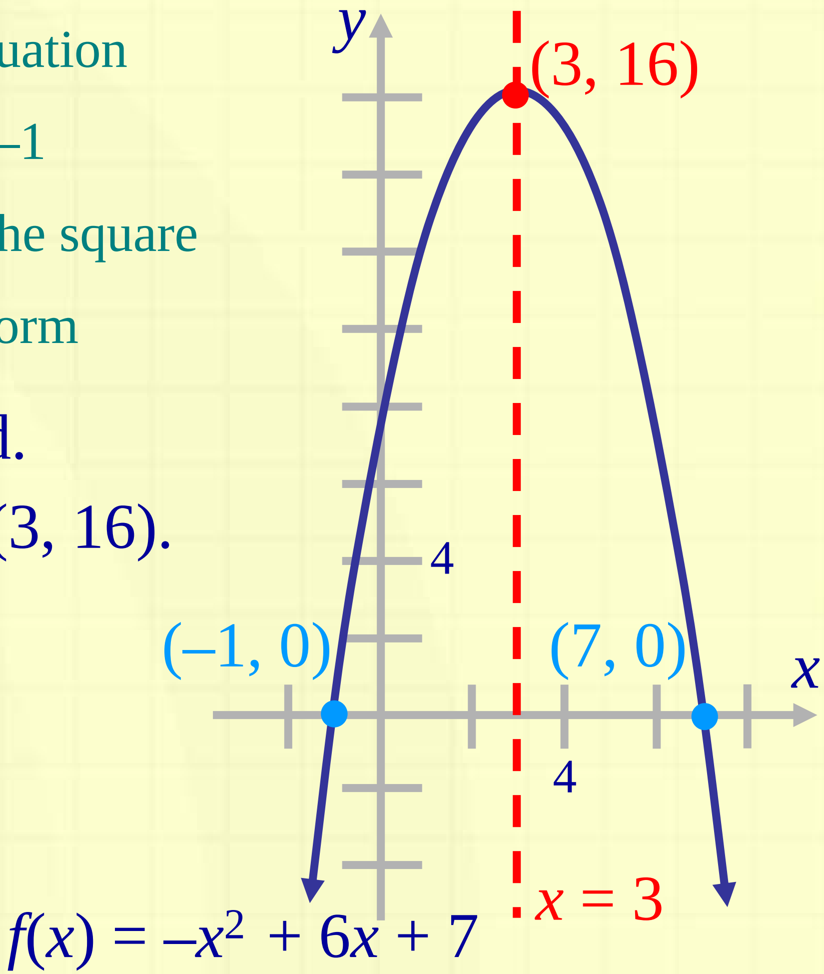
Find the x-intercepts by solving

$-x^2 + 6x + 7 = 0$.

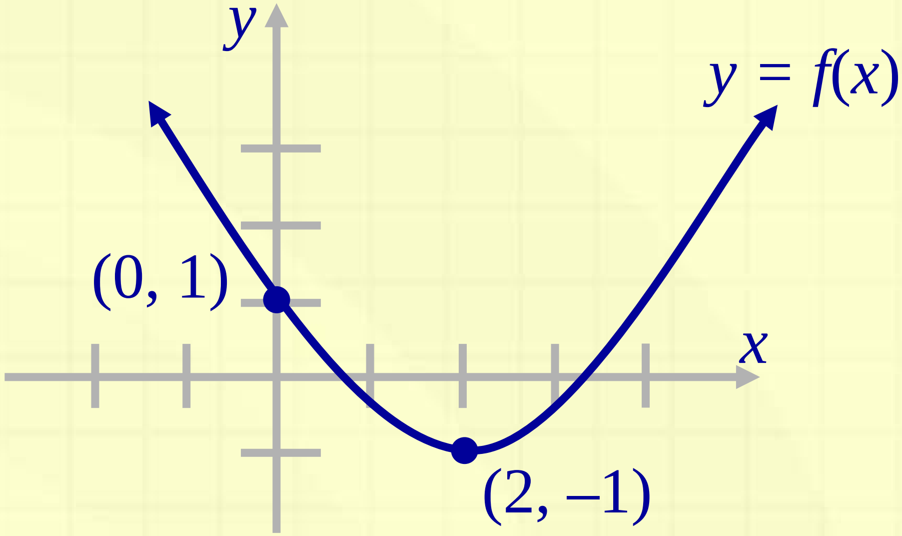
$(-x + 7)(x + 1) = 0$ factor

$x = 7,\ x = -1$

x-intercepts $(7, 0)$, $(-1, 0)$



Example: Find an equation for the parabola with vertex (2, –1) passing through the point (0, 1).



$f(x) = a(x - h)^2 + k$ standard form

$f(x) = a(x - 2)^2 + (-1)$ vertex $(2, -1) = (h, k)$

Since (0, 1) is a point on the parabola: $f(0) = a(0 - 2)^2 - 1$

$1 = 4a - 1$ and $a = \dfrac{1}{2}$

$$f(x) = \frac{1}{2}(x-2)^2 - 1 \rightarrow f(x) = \frac{1}{2}x^2 - 2x + 1$$

Vertex of a Parabola

The *vertex* of the graph of $f(x) = ax^2 + bx + c \ (a \neq 0)$ is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$

Example: Find the vertex of the graph of $f(x) = x^2 - 10x + 22$.

$f(x) = x^2 - 10x + 22$ original equation

$a = 1, b = -10, c = 22$

At the vertex, $x = \dfrac{-b}{2a} = \dfrac{-10}{2(1)} = 5$

$$f\left(\frac{-b}{2a}\right) = f(5) = 5^2 - 10(5) + 22 = -3$$

So, the vertex is (5, -3).

Example: A basketball is thrown from the free throw line from a height of six feet. What is the maximum height of the ball if the path of the ball is: $y = -\frac{1}{9}x^2 + 2x + 6.$

The path is a parabola opening downward.
The maximum height occurs at the vertex.

$$y = \frac{-1}{9}x^2 + 2x + 6 \rightarrow a = \frac{-1}{9},\ b = 2$$

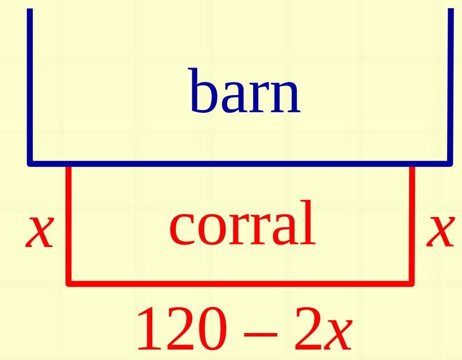
At the vertex, $x = \frac{-b}{2a} = 9.$

$$f\left(\frac{-b}{2a}\right) = f(9) = 15$$

So, the vertex is (9, 15).

The maximum height of the ball is 15 feet.

Example: A fence is to be built to form a rectangular corral along the side of a barn 65 feet long. If 120 feet of fencing are available, what are the dimensions of the corral of maximum area?



Let x represent the width of the corral and $120 - 2x$ the length.

$$\text{Area} = A(x) = (120 - 2x)x = -2x^2 + 120x$$

The graph is a parabola and opens downward.

The maximum occurs at the vertex where $x = \dfrac{-b}{2a}$,

$$a = -2 \text{ and } b = 120 \rightarrow x = \frac{-b}{2a} = \frac{-120}{-4} = 30.$$

$$120 - 2x = 120 - 2(30) = 60$$

The maximum area occurs when the width is 30 feet and the length is 60 feet.

Chapter 2 – Section 4

Add, Subtract, Multiply Polynomials

A polynomial of two terms is a binomial.

$$7xy^2 + 2y$$

A polynomial of three terms is a trinomial.

$$8x^2 + 12xy + 2y^2$$

The *leading coefficient* of a polynomial is the coefficient of the variable with the largest exponent.

The *constant term* is the term without a variable.

The degree is 3.

$$\boxed{6}x^{\boxed{3}} - 2x^2 + 8x + \boxed{15}$$

The leading coefficient is 6. The constant term is 15.

The *degree of a polynomial* is the greatest of the degrees of any of its terms. The degree of a term is the sum of the exponents of the variables.

Examples: $3y^2 + 5x + 7$ degree 2

$21x^5y + 3x^3 + 2y^2$ degree 6

Common polynomial functions are named according to their degree.

Function	Equation	Degree
linear	$f(x) = mx + b$	one
quadratic	$f(x) = ax^2 + bx + c, a \neq 0$	two
cubic	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$	three

To add polynomials, combine like terms.

Examples: 1. Add $(5x^3 + 6x^2 + 3) + (3x^3 - 12x^2 - 10)$.
Use a horizontal format.

$$
\begin{aligned}
& (5x^3 + 6x^2 + 3) + (3x^3 - 12x^2 - 10) \\
&= (5x^3 + 3x^3) + (6x^2 - 12x^2) + (3 - 10) \\
&= 8x^3 - 6x^2 - 7
\end{aligned}
$$

Rearrange and group like terms.

Combine like terms.

2. Add $(6x^3 + 11x - 21) + (2x^3 + 10 - 3x) + (5x^3 + x - 7x^2 + 5)$.
Use a vertical format.

$$
\begin{array}{rrrr}
6x^3 & & +\ 11x & -\ 21 \\
2x^3 & & -\ 3x & +\ 10 \\
5x^3 & -\ 7x^2 & +\ x & +\ 5 \\
\hline
13x^3 & -\ 7x^2 & +\ 9x & -\ 6
\end{array}
$$

Arrange terms of each polynomial in descending order with like terms in the same column.

Add the terms of each column.

The *additive inverse* of the polynomial $x^2 + 3x + 2$ is $-(x^2 + 3x + 2)$.

This is equivalent to the additive inverse of each of the terms.

$$-(x^2 + 3x + 2) = -x^2 - 3x - 2$$

To subtract two polynomials, add the additive inverse of the second polynomial to the first.

Example: Add $(4x^2 - 5xy + 2y^2) - (-x^2 + 2xy - y^2)$.

$$\begin{aligned} &(4x^2 - 5xy + 2y^2) - (-x^2 + 2xy - y^2) \\ &= (4x^2 - 5xy + 2y^2) + (x^2 - 2xy + y^2) && \text{Rewrite the subtraction as the addition of the additive inverse.} \\ &= (4x^2 + x^2) + (-5xy - 2xy) + (2y^2 + y^2) && \text{Rearrange and group like terms.} \\ &= 5x^2 - 7xy + 3y^2 && \text{Combine like terms.} \end{aligned}$$

Let $P(x) = 2x^2 - 3x + 1$ and $R(x) = -x^3 + x + 5$.

Examples: 1. Find $P(x) + R(x)$.

$$\begin{aligned} P(x) + R(x) &= (2x^2 - 3x + 1) + (-x^3 + x + 5) \\ &= -x^3 + 2x^2 + (-3x + x) + (1 + 5) \\ &= -x^3 + 2x^2 - 2x + 6 \end{aligned}$$

2. If $D(x) = P(x) - R(x)$, find $D(-2)$.

$$\begin{aligned} P(x) - R(x) &= (2x^2 - 3x + 1) - (-x^3 + x + 5) \\ &= (2x^2 - 3x + 1) + (x^3 - x - 5) \\ &= x^3 + 2x^2 - 4x - 4 \\ D(-2) &= (-2)^3 + 2(-2)^2 - 4(-2) - 4 \\ &= 4 \end{aligned}$$

To *multiply a polynomial by a monomial*, use the distributive property and the rule for multiplying exponential expressions.

Examples: 1. Multiply: $2x(3x^2 + 2x - 1)$.

$$= 2x(3x^2) + 2x(2x) + 2x(-1)$$

$$= 6x^3 + 4x^2 - 2x$$

2. Multiply: $-3x^2y(5x^2 - 2xy + 7y^2)$.

$$= -3x^2y(5x^2) - 3x^2y(-2xy) - 3x^2y(7y^2)$$

$$= -15x^4y + 6x^3y^2 - 21x^2y^3$$

To multiply two polynomials, apply the distributive property.

Example: Multiply: $(x - 1)(2x^2 + 7x + 3)$.

$$= (x-1)(2x^2) + (x-1)(7x) + (x-1)(3)$$

$$= 2x^3 - 2x^2 + 7x^2 - 7x + 3x - 3$$

$$= 2x^3 + 5x^2 - 4x - 3$$

Two polynomials can also be multiplied using a vertical format.

Example:

$$\begin{array}{rrrrl}
 & 2x^2 & + 7x & + 3 & \\
 & & x & - 1 & \\
\hline
 & -2x^2 & - 7x & - 3 & \text{Multiply } -1(2x^2 + 7x + 3). \\
2x^3 & + 7x^2 & + 3x & & \text{Multiply } x(2x^2 + 7x + 3). \\
\hline
2x^3 & + 5x^2 & - 4x & - 3x & \text{Add the terms in each column.}
\end{array}$$

To multiply two binomials use a method called FOIL, which is based on the distributive property. The letters of FOIL stand for First, Outer, Inner, and Last.

1. Multiply the first terms.
2. Multiply the outer terms.
3. Multiply the inner terms.
4. Multiply the last terms.
5. Add the products.
6. Combine like terms.

Examples: 1. Multiply: $(2x + 1)(7x - 5)$.

$$= \overbrace{2x(7x)}^{\text{First}} + \overbrace{2x(-5)}^{\text{Outer}} + \overbrace{(1)(7x)}^{\text{Inner}} + \overbrace{(1)(-5)}^{\text{Last}}$$

$$= 14x^2 - 10x + 7x - 5$$

$$= 14x^2 - 3x - 5$$

2. Multiply: $(5x - 3y)(7x + 6y)$.

$$= \overbrace{5x(7x)}^{\text{First}} + \overbrace{5x(6y)}^{\text{Outer}} + \overbrace{(-3y)(7x)}^{\text{Inner}} + \overbrace{(-3y)(6y)}^{\text{Last}}$$

$$= 35x^2 + 30xy - 21yx - 18y^2$$

$$= 35x^2 + 9xy - 18y^2$$

The multiply the sum and difference of two terms, use this pattern:

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$

$$= a^2 - b^2$$

square of the second term

square of the first term

Examples: 1. $(3x + 2)(3x - 2)$

$= (3x)^2 - (2)^2$

$= 9x^2 - 4$

2. $(x + 1)(x - 1)$

$= (x)^2 - (1)^2$

$= x^2 - 1$

To square a binomial, use this pattern:

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

square of the first term

twice the product of the two terms

square of the last term

Examples: 1. Multiply: $(2x - 2)^2$.

$$= (2x)^2 + 2(2x)(-2) + (-2)^2$$

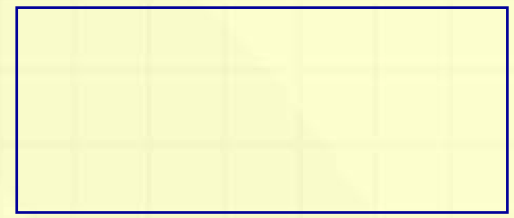
$$= 4x^2 - 8x + 4$$

2. Multiply: $(x + 3y)^2$.

$$= (x)^2 + 2(x)(3y) + (3y)^2$$

$$= x^2 + 6xy + 9y^2$$

Example: The length of a rectangle is $(x + 5)$ ft. The width is $(x – 6)$ ft. Find the area of the rectangle in terms of the variable x.



$$A = L \cdot W = \text{Area}$$

$$L = (x + 5) \text{ ft}$$

$$W = (x – 6) \text{ ft}$$

$$A = (x + 5)(x – 6) = x^2 – 6x + 5x – 30$$

$$= x^2 – x – 30$$

The area is $(x^2 – x – 30)$ ft^2.

