# Chapter 3 – Sections 1 & 3

# Matrices

A **matrix** is a rectangular array of real numbers.
Matrix $A$ has 2 horizontal **rows** and 3 vertical **columns.**

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 7 & -1 & 0.5 \end{bmatrix}$$

Each **entry** can be identified by its position in the matrix.

7 is in Row 2 Column 1.

-2 is in Row 1 Column 3.

A matrix with $m$ rows and $n$ columns is of **order** $m \times n$.

$A$ is of order $2 \times 3$.

If $m = n$ the matrix is said to be **square** of order $n$.

  

**Examples**: Find the order of each matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 2 & 1 & 4 \\ 1 & 1 & 6 & 2 \end{bmatrix}$$

$A$ has *three* rows and *four* columns.
The order of $A$ is $3 \times 4$.

$$B = \begin{bmatrix} 2 & 5 & 2 & -1 & 0 \end{bmatrix}$$

$B$ has *one* row and *five* columns.
The order of $B$ is $1 \times 5$.
$B$ is called a **row matrix**.

$$C = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$C$ is a $2 \times 2$ *square* matrix.

  

An $m \times n$ matrix can be written

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m1} & a_{m1} & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same order and $a_{ij} = b_{ij}$ for every $i$ and $j$.

For example, $\begin{bmatrix} 0.5 & \sqrt{9} \\ \frac{1}{4} & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 3 \\ 0.25 & 7 \end{bmatrix}$ since both matrices are of order $2 \times 2$ and all corresponding entries are equal.

  

To **add** matrices:

1. Check to see if the matrices have the same order.
2. Add corresponding entries.

**Example**: Find the sums $A + B$ and $B + C$.

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 1 \\ 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 0 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

$A$ has order $3 \times 2$ and $B$ has order $2 \times 3$. So they cannot be added. $C$ has order $2 \times 3$ and can be added to $B$.

$$B + C = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 0 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 2 & 2 & 1 \end{bmatrix}$$

  

To **subtract** matrices:

1. Check to see if the matrices have the same order.
2. Subtract corresponding entries.

**Example**: Find the differences $A - B$ and $B - C$.

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 4 & -5 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$A$ and $B$ are both of order $2 \times 2$ and can be subtracted.

$$A - B = \begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ -2 & 6 \end{bmatrix}$$

Since $B$ is of order $2 \times 2$ and $C$ is of order $3 \times 2$, they cannot be subtracted.

  

If $A = [a_{ij}]$ is an $m \times n$ matrix and $c$ is a **scalar** (a real number), then the $m \times n$ matrix $cA = [ca_{ij}]$ is the **scalar multiple** of $A$ by $c$.

Example: Find $2A$ and $-3A$ for $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 0 \\ 2 & 7 & 2 \end{bmatrix}$.

$$2A = \begin{bmatrix} 2(2) & 2(5) & 2(-1) \\ 2(3) & 2(4) & 2(0) \\ 2(2) & 2(7) & 2(2) \end{bmatrix} = \begin{bmatrix} 4 & 10 & -2 \\ 6 & 8 & 0 \\ 4 & 14 & 4 \end{bmatrix}$$

$$-\frac{1}{3}A = \begin{bmatrix} -3(2) & -3(5) & -3(-1) \\ -3(3) & -3(4) & -3(0) \\ -3(2) & -3(7) & -3(2) \end{bmatrix} = \begin{bmatrix} -6 & -15 & 3 \\ -9 & -12 & 0 \\ -6 & -21 & -6 \end{bmatrix}$$

  

**Example**: Calculate the value of $3A - 2B + C$ with

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$3A - 2B + C = 3\begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 \\ 9 & 15 \\ 12 & -6 \end{bmatrix} - \begin{bmatrix} 10 & 4 \\ 2 & 0 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 8 & 15 \\ 9 & -5 \end{bmatrix}$$

  

An **augmented matrix** and a **coefficient matrix** are associated with each system of linear equations.

For the system $\begin{cases} 2x + 3y - z = 12 \\ x - 8y = 16 \end{cases}$

The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 12 \\ 1 & -8 & 0 & 16 \end{array}\right]$.

The coefficient matrix is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -8 & 0 \end{bmatrix}$.

## **Elementary Row Operations**.

1. Interchange two rows of a matrix.
2. Multiply a row of a matrix by a nonzero constant.
3. Add a multiple of one row of a matrix to another.

A sequence of elementary row operations transforms the augmented matrix of a system into the augmented matrix of another system with the same solutions as the original system.

In this case we say the augmented matrices are **row equivalent**.

  

**Example**: Apply the elementary row operation $R_1 \leftrightarrow R_2$ to the augmented matrix of the system $\begin{cases} x + 2y = 8 \\ 3x - y = 10 \end{cases}$.

| Row Operation | Augmented Matrix | System |
|---|---|---|
| | $\left[\begin{array}{cc:c} 1 & 2 & 8 \\ 3 & -1 & 10 \end{array}\right]$ | $\begin{cases} x + 2y = 8 \\ 3x - y = 10 \end{cases}$ |
| $R_1 \leftrightarrow R_2$ | $\downarrow$ | $\downarrow$ |
| | $\left[\begin{array}{cc:c} 3 & -1 & 10 \\ 1 & 2 & 8 \end{array}\right]$ | $\begin{cases} 3x - y = 10 \\ x + 2y = 8 \end{cases}$ |

Note that the two systems are equivalent.

  

**Example**: Apply the elementary row operation $3R_2$ to the augmented matrix of the system $\begin{cases} x+2y=8 \\ 3x-y=10 \end{cases}$.

| Row Operation | Augmented Matrix | System |
|---|---|---|
| | $\left[\begin{array}{cc:c} 1 & 2 & 8 \\ 3 & -1 & 10 \end{array}\right]$ | $\begin{cases} x+2y=8 \\ 3x-y=10 \end{cases}$ |
| $3R_2$ | $\downarrow$ | $\downarrow$ |
| | $\left[\begin{array}{cc:c} 3 & 6 & 24 \\ 9 & -3 & 30 \end{array}\right]$ | $\begin{cases} 3x+6y=24 \\ 9x-3y=30 \end{cases}$ |

  

**Example**: Apply the row operation $-3R_1 + R_2$ to the augmented matrix of the system $\begin{cases} x + 2y = 8 \\ 3x - y = 10 \end{cases}$.

| Row Operation | Augmented Matrix | System |
|---|---|---|
| | $\left[\begin{array}{cc:c} 1 & 2 & 8 \\ 3 & -1 & 10 \end{array}\right]$ | $\begin{cases} x + 2y = 8 \\ 3x - y = 10 \end{cases}$ |
| $-3R_1 + R_2$ | $\downarrow$ | $\downarrow$ |
| | $\left[\begin{array}{cc:c} 1 & 2 & 8 \\ 0 & -7 & -14 \end{array}\right]$ | $\begin{cases} x + 2y = 8 \\ -7y = -14 \end{cases}$ |

  