

Lab 2: Limits on Borrowing - Revisited

We will revisit the loan formula (below) from lab 1. If you pay at least some amount each month, no matter how small, will you be able to eventually pay off your loan, given enough time? Intuition would suggest the answer is yes; eventually any loan will get paid off.

The Loan Formula

$$S + P\left(1 + \frac{r}{m}\right)^{mt} + R\left[\frac{\left(1 + \frac{r}{m}\right)^{mt} - 1}{\frac{r}{m}}\right] = 0$$

Explanation of Loan Formula

Symbol	Usual Meaning	Mortgage/Loan Specific Meaning
S	Future Value	\$0
P	Present Value or Principal	Amount (usually positive) you borrow
R	Periodic Payment	Amount (usually negative) you pay ea. month
r	Interest Rate	APR or annual percentage rate (in decimal)
t	Time	Number of years of the mortgage/loan
m	Compounding Rate	How often interest is computed (usually monthly)

A definite answer, however, is going to require us to analyze the loan formula again. Be prepared to be surprised!

Let's break the loan formula into two competing parts. One part of the loan formula deal with the growing amount of interest the bank makes from your loan over time. Let's label this part of the formula $G(t)$:

$$G(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

The other main part of the loan formula deals with how much you decrease your debt over time by making regular payments. Call this part $D(t)$:

$$D(t) = R\left[\frac{\left(1 + \frac{r}{m}\right)^{mt} - 1}{\frac{r}{m}}\right]$$

To pay off any loan, $D(t)$ must be increasing at a greater rate than $G(t)$. In other words, the amount you are paying $D(t)$ has to outpace the growth of the interest on the principal $G(t)$ the bank is adding to the loan. If so, then eventually, the amount you pay $D(t)$ will catch up to $G(t)$, the amount the bank is earning from your loan, and your loan will be paid off.

Notice that our discussion involves *rates of change* of $G(t)$ and $D(t)$. What does that phrase suggest we do? Analyze this problem scenario by taking derivatives of $G(t)$ and $D(t)$. Let's do some calculus!

Step One: Compute a Derivative

Compute $G'(t)$, the derivative of G with respect to time t . Show your work and answer on a separate sheet of paper.

Step Two: Compute a Derivative (Again)

Compute $D'(t)$, the derivative of D with respect to time t . Show your work and answer on a separate sheet of paper.

Some Hints for Step One and Step Two:

1. The letter t is the only variable in these functions. All those other letters are constants. You may need to use the constant rule when differentiating.
2. These functions are exponential functions. (What is the base of each function?) You will need to use the exponential rule when differentiating.

Step Three: Set Up an Inequality

Based on our discussion above, which situation do we want to be true?

$$G'(t) < D'(t) \quad \text{or} \quad D'(t) < G'(t)$$

Circle the correct answer.

Step Four: Do the Algebra

Set up the correct inequality and solve for R (your monthly payment). Note: the inequality may look frightening, but be patient, and a whole bunch of terms will cancel out. Thank goodness for algebra! Show your work and solution on a separate sheet of paper.

Step Five: Interpretation

Interpret your result. Is there a minimum amount that you must make on your monthly payment R to pay off the loan? If so, what is that amount? Answer in a complete sentence on a separate sheet of paper.

Observe that your answer is very similar to the limit computed in lab 1. In other words, the maximum you can borrow (lab 1) is related to the minimum required monthly payment (lab 2).

Step Six: An Applied Example

Use the value of r from your lab 1 to compute the minimum required monthly payment R to pay off a loan of \$10,000. Record your work and answer a separate sheet of paper.