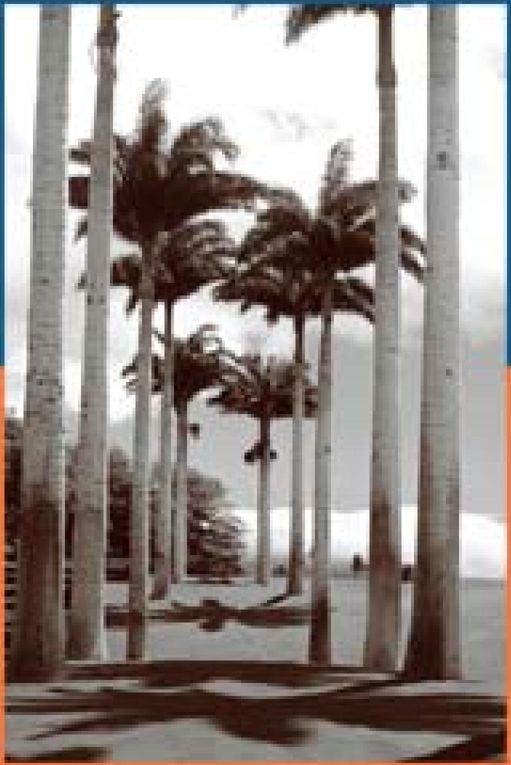
COLLEGE ALGEBRA

with Modeling and Visualization



GARY ROCKSWOLD

THIRD EDITION

1.3 Functions and Their Representations

- Learn function notation
- Represent a function four different ways
- Define a function formally
- Identify the domain and range of a function
- Use calculators to represent functions (optional)
- Identify functions

Idea Behind a Function

- Recall that a relation is a set of ordered pairs (x,y) .
- If we think of values of x as being inputs and values of y as being outputs, a function is a relation such that
  - for each input there is exactly one output.

This is symbolized by *output* = *f*(input) or

$$y = f(x)$$

Function Notation

- $y = f(x)$
  - Is pronounced "y is a function of x."
  - Means that given a value of x (input), there is exactly one corresponding value of y (output).
  - x is called the independent variable as it represents inputs, and y is called the dependent variable as it represents outputs.
  - Note that: $f(x)$ is NOT f multiplied by x. f is NOT a variable, but the name of a function (the name of a relationship between variables).

Domain and Range of a Function

- The set of all meaningful inputs is called the DOMAIN of the function.
- The set of corresponding outputs is called the RANGE of the function.

Formal Definition of a Function

- A **function** is a relation in which each element of the domain corresponds to exactly one element in the range.

Example 1

- Suppose a car travels at 70 miles per hour. Let y be the distance the car travels in x hours. Then $y = 70x$.
- Since for each value of x (that is the time in hours the car travels) there is just one corresponding value of y (that is the distance traveled), y is a function of x and we write

$$y = f(x) = 70x$$

- Evaluate f(3) and interpret.
  - $f(3) = 70(3) = 210$. This means that the car travels 210 miles in 3 hours.

Example 2

Given the following data, is y a function of x?

Input x	3	4	8
Output y	6	6	– 5

Note: The data in the table can be written as the set of ordered pairs {(3,6), (4,6), (8, –5)}.

Yes, y is a function of x, because for each value of x, there is just one corresponding value of y. Using function notation we write $f(3) = 6$; $f(4) = 6$; $f(8) = -5$.

Example 3

- Undergraduate Classification at Study-Hard University (SHU) is a function of Hours Earned. We can write this in function notation as $C = f(H)$.
- Why is C a function of H?
  - For each value of H there is exactly one corresponding value of C.
  - In other words, for each input there is exactly one corresponding output.

$C = f(H)$

- Classification of Students at SHU

From Catalogue

```
No student may be classified
as a sophomore until after
earning at least 30 semester
hours.

No student may be classified
as a junior until after
earning at least 60 hours.

No student may be classified
as a senior until after
earning at least 90 hours.
```

- Evaluate $f(20)$
  - $f(20)$ = **Freshman**
- Evaluate $f(30)$
  - $f(30)$ = **Sophomore**
- Evaluate $f(0)$
  - $f(0)$ = **Freshman**
- Evaluate $f(61)$
  - $f(61)$ = **Junior**
- What is the domain of f?
- What is the **range** of f?

$C = f(H)$

- Domain of f is the set of non-negative integers

$$\{0,1,2,3,4\ldots\}$$

  - Alternatively, some individuals say the domain is the set of positive rational numbers, since technically one could earn a fractional number of hours if they transferred in some quarter hours. For example, 4 quarter hours = 2 2/3 semester hours.
  - Some might say the domain is the set of non-negative real numbers $[0,\infty)$, but this set includes irrational numbers. It is impossible to earn an irrational number of credit hours. For example, one could not earn $\sqrt{2}$ hours.

- Range of *f* is {Fr, Soph, Jr, Sr}

Questions: Identifying Functions

- Referring to the previous example concerning SHU, is hours earned a function of classification? That is, is $H = f(C)$? Explain why or why not.
- Is classification a function of years spent at SHU? Why or why not?
- Given $x = y^2$, is y a function of x? Why or why not?
- Given $x = y^2$, is x a function of y? Why or why not?
- Given $y = x^2 - 2$, is y a function of x? Why, why not?

Answers

- Is hours earned a function of classification? That is, is $H = f(C)$?
- That is, for each value of C is there just one corresponding value of H?
  - No. One example is
    - if C = Freshman, then H could be 3 or 10 (or lots of other values for that matter)

Answers continued

- Is classification a function of years spent at SHU? That is, is $C = f(Y)$?
- That is, for each value of Y is there just one corresponding value of C?
  - No. One example is
    - if $Y = 4$, then C could be Sr. or Jr. It could be Jr if a student was a part time student and full loads were not taken.

Answers continued

- Given $x = y^2$, is y a function of x?
- That is, given a value of x, is there just one corresponding value of y?
  - No, if $x = 4$, then $y = 2$ or $y = -2$.

Answers continued

- Given $x = y^2$, is x a function of y?
- That is, given a value of y, is there just one corresponding value of x?
  - Yes, given a value of y, there is just one corresponding value of x, namely y^2.

Answers continued

- Given $y = x^2 - 2$, is y a function of x?
- That is, given a value of x, is there just one corresponding value of y?
  - Yes, given a value of x, there is just one corresponding value of y, namely $x^2 - 2$.

Five Ways to Represent a Function (page 31)

- Verbally
- Numerically
- Diagrammaticly
- Symbolically
- Graphically

$C = f(H)$ (Referring to previous SHU example)

- Verbal Representation.
  - If you have less than 30 hours, you are a freshman.
  - If you have 30 or more hours, but less than 60 hours, you are a sophomore.
  - If you have 60 or more hours, but less than 90 hours, you are a junior.
  - If you have 90 or more hours, you are a senior.

$C = f(H)$

Numeric Representation

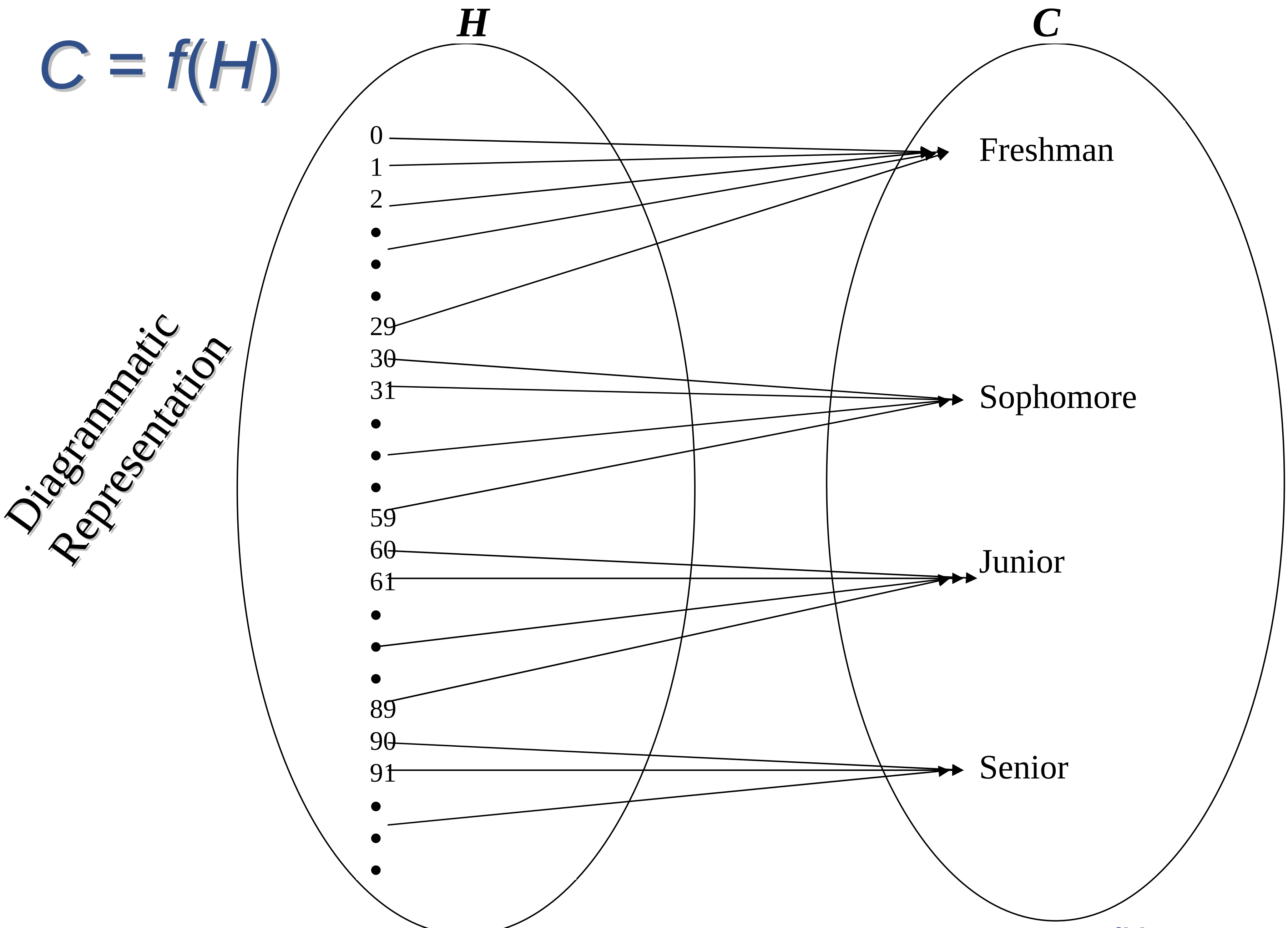
H	*C*
0	Freshman
1	Freshman
? ? ? ?	
29	Freshman
30	Sophomore
31	Sophomore
? ? ?	
59	Sophomore
60	Junior
61	Junior
? ? ?	
89	Junior
90	Senior
91	Senior
? ? ?	

$C = f(H)$

Symbolic Representation

$$C = f(H) = \begin{cases} \text{Freshman} & \text{if } 0 \le H < 30 \\ \text{Sopho} & \text{if } 30 \le H < 60 \\ \text{Junior} & \text{if } 60 \le H < 90 \\ \text{Senior} & \text{if } H \ge 90 \end{cases}$$

$C = f(H)$



Diagrammatic Representation
H
C
0
1
2
29
30
31
59
60
61
89
90
91
Freshman
Sophomore
Junior
Senior

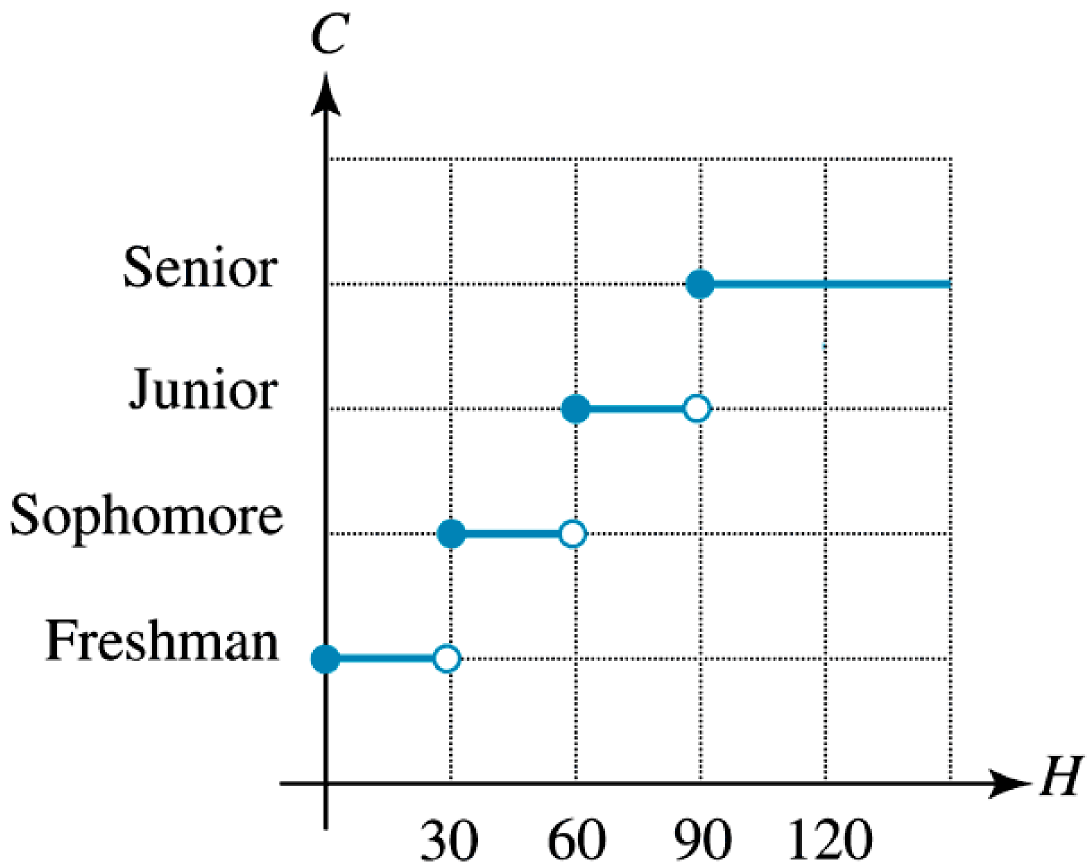
$C = f(H)$

Graphical Representation

Note that in this graph the domain is considered to be $[0, \infty)$

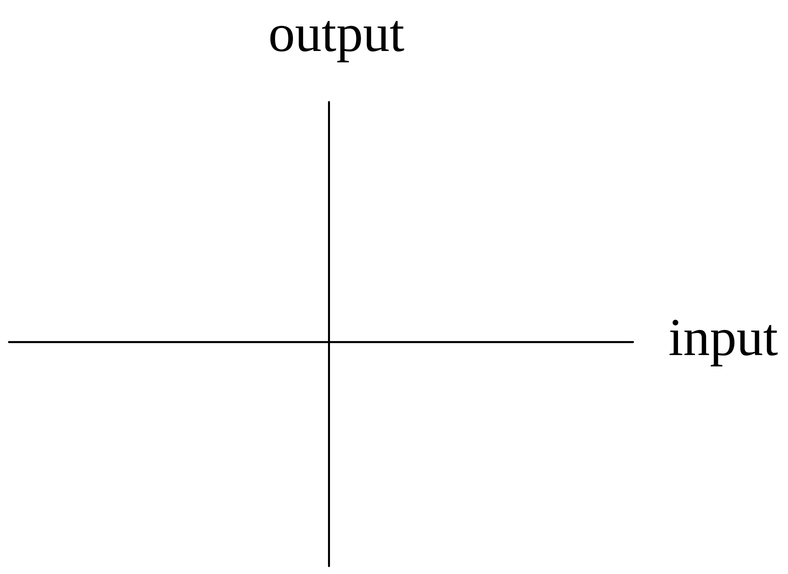
instead of

$\{0,1,2,3\ldots\}$



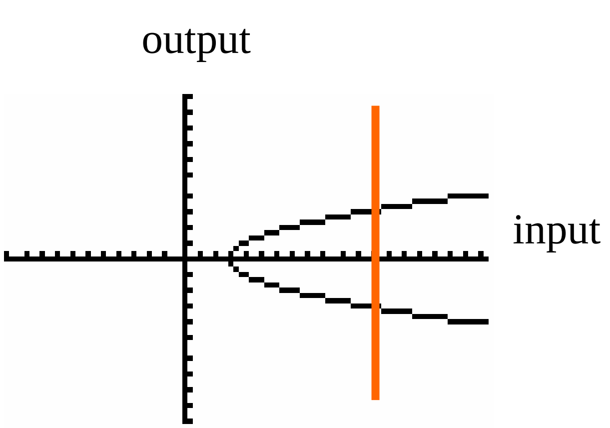
Some Notes on Graphical Representation

- Inputs are typically graphed on the horizontal axis and outputs are typically graphed on the vertical axis.



Notes on Graphical Representation Continued

- Vertical line test (p 39). To determine if a graph represents a function, simply visualize vertical lines in the *xy*-plane. If each vertical line intersects a graph at no more than one point, then it is the graph of a function.
  Why does this work?



1.4 Types of Functions and Their Rates of Change

- Identify and use constant and linear functions
- Interpret slope as a rate of change
- Identify and use nonlinear functions
- Recognize linear and nonlinear data
- Use and interpret average rate of change
- Calculate the difference quotient

Constant Function

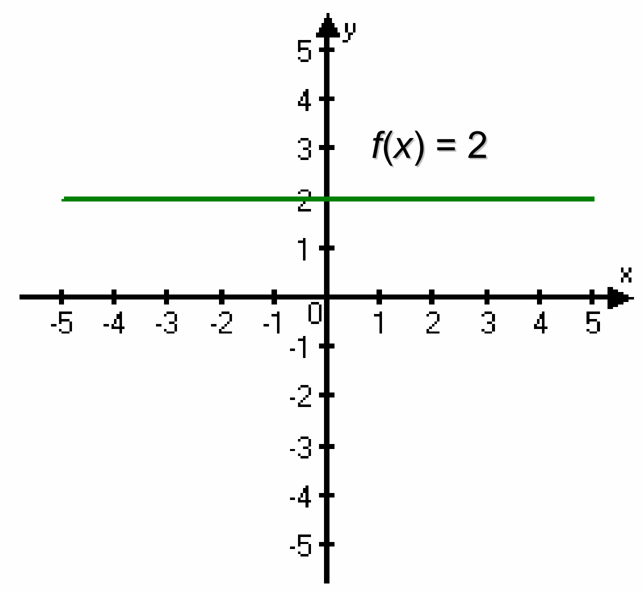
- A function *f* represented by *f*(*x*) = *b*, where *b* is a constant (fixed number), is a constant function.

Examples:

$$f(x) = 2$$

$$f(x) = \frac{-1}{2}$$

$$f(x) = \sqrt{2}$$



Note: Graph of a constant function is a horizontal line.

Linear Function

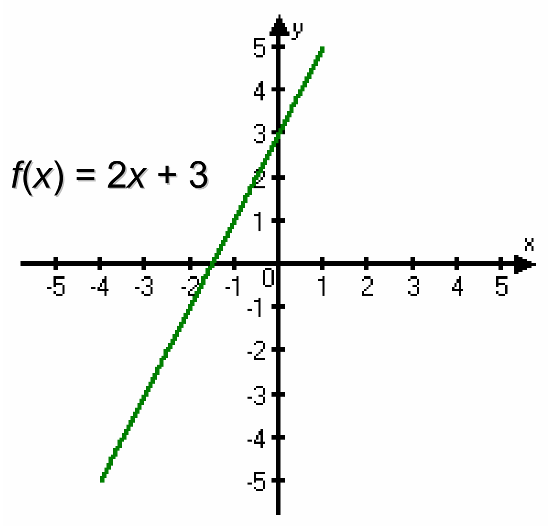
- A function f represented by $f(x) = ax + b$, where a and b are constants, is a linear function.

Examples:

$$f(x) = 2x + 3 \qquad (\text{Note: } a = 2 \text{ and } b = 3)$$

$$f(x) = -5x - \frac{1}{2} \qquad \left(\text{Note: } a = -5 \text{ and } b = -\frac{1}{2}\right)$$

$$f(x) = 2 \qquad (\text{Note: } a = 0 \text{ and } b = 2)$$



Note that a $f(x) = 2$ is both a linear function and a constant function.
A constant function is a special case of a linear function.

Rate of Change of a Linear Function

- Table of values for $f(x) = 2x + 3$.

x	y
–2	–1
–1	1
0	3
1	5
2	7
3	9

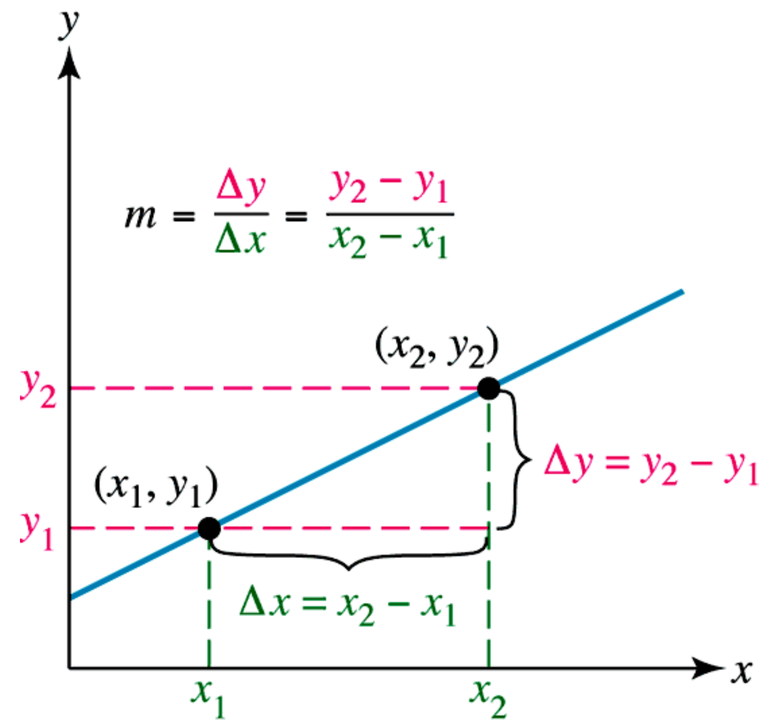
Note throughout the table, as x increases by 1 unit, y increases by 2 units. In other words, the RATE OF CHANGE of y with respect to x is constantly 2 throughout the table. Since the rate of change of y with respect to x is constant, the function is LINEAR. Another name for rate of change of a linear function is SLOPE.

Slope of Line

- The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

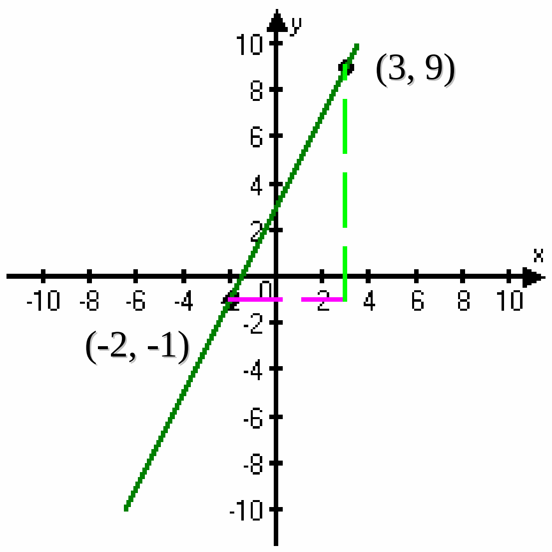
where $x_1 \neq x_2$



Example of Calculation of Slope

- Find the slope of the line passing through the points (–2, –1) and (3, 9).

$$m = \frac{\Delta y}{\Delta x} = \frac{9-(-1)}{3-(-2)} = \frac{10}{5} = 2$$

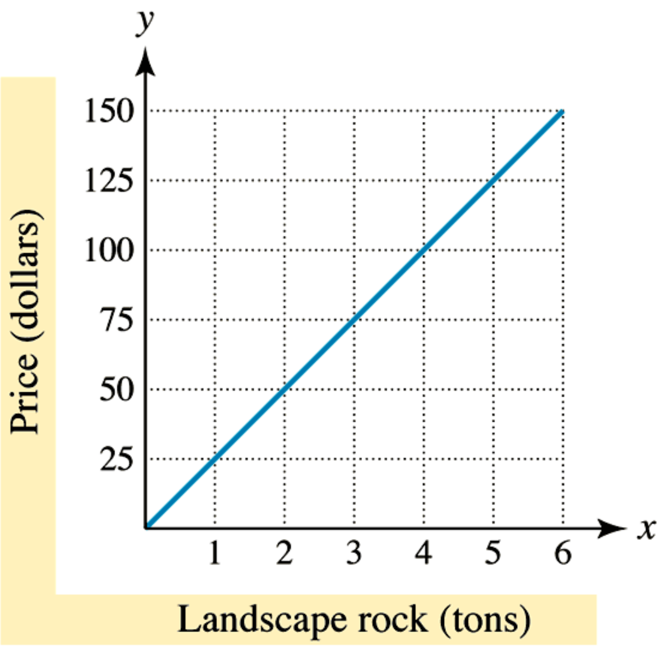


The slope being 2 means that for each unit x increases, the corresponding increase in *y* is 2. The rate of change of *y* with respect to x is 2/1 or 2.

Another Example of a Linear Function

- The table and corresponding graph show the price *y* of *x* tons of landscape rock.

X (tons)	*y* (price in dollars)
1	25
2	50
3	75
4	100



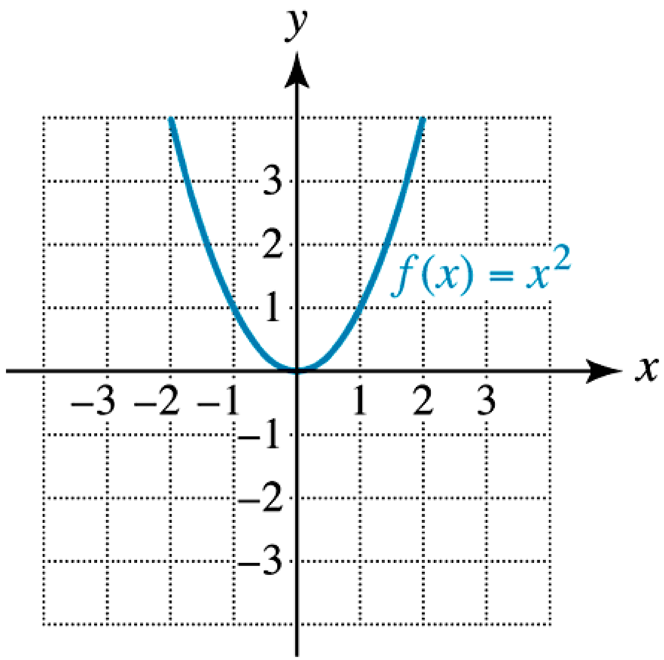
y is a linear function of *x* and the slope is $\frac{\Delta y}{\Delta x} = \frac{50-25}{2-1} = 25$

The rate of change of price *y* with respect to tonage *x* is 25 to 1. This means that for an increase of 1 ton of rock the price increases by $25.

Example of a Nonlinear Function

- Table of values for $f(x) = x^2$

x	y
0	0
1	1
2	4



Note that as x increases from 0 to 1, y increases by 1 unit; while as x increases from 1 to 2, y increases by 3 units. 1 does not equal 3. This function does NOT have a CONSTANT RATE OF CHANGE of y with respect to x, so the function is NOT LINEAR. Note that the graph is not a line.

Average Rate of Change

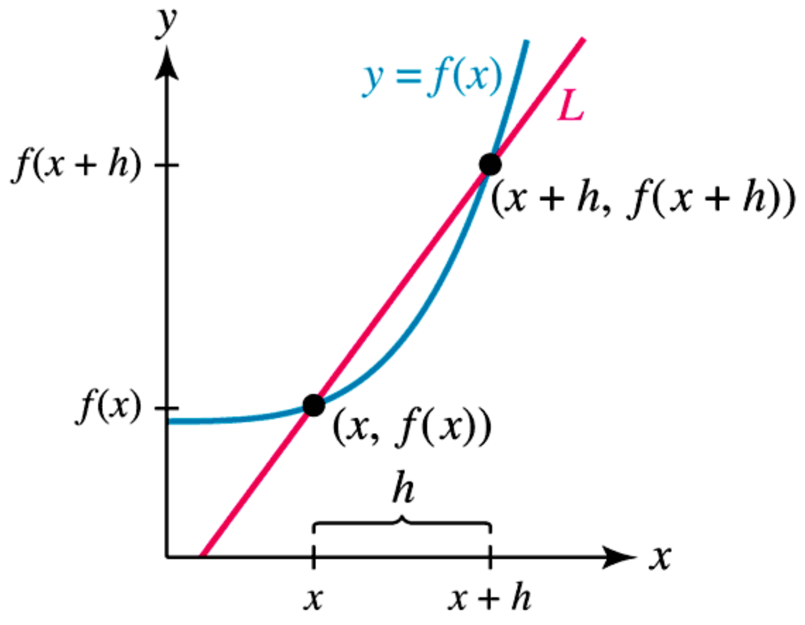
- Let (x_1, y_1) and (x_2, y_2) be distinct points on the graph of a function *f*. The average rate of change of *f* from x_1 to x_2 is $\dfrac{y_2 - y_1}{x_2 - x_1}$

Note that the average rate of change of f from x_1 to x_2 is the slope of the line passing through (x_1, y_1) and (x_2, y_2)

The Difference Quotient

- The difference quotient of a function *f* is an expression of the form $\dfrac{f(x+h)-f(x)}{h}$ where *h* is not 0.

Note that a difference quotient is actually an average rate of change.



Example of Calculating a Difference Quotient

- Let $f(x) = x^2 + 3x$. Find the difference quotient of *f* and simplify the result.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+3(x+h)-(x^2+3x)}{h} =$$

$$\frac{(x^2+2xh+h^2)+3x+3h-x^2-3x}{h} = \frac{2xh+h^2+3h}{h} =$$

$$\frac{h(2x+h+3)}{h} = 2x+h+3$$