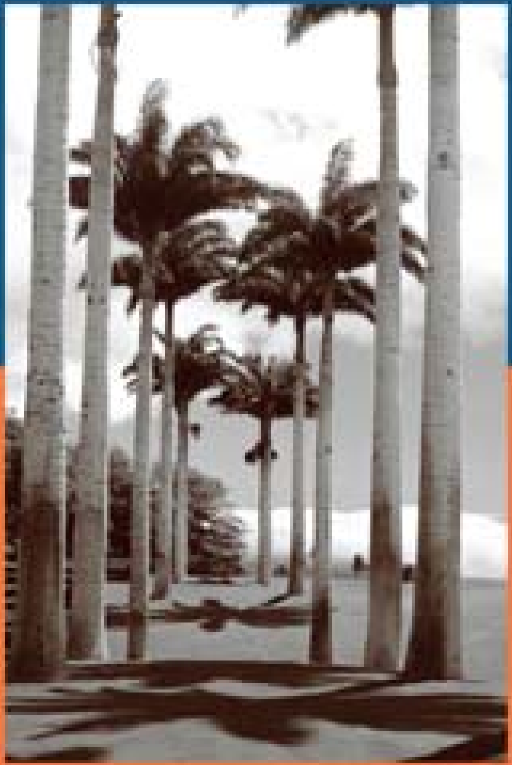
COLLEGE ALGEBRA

with Modeling and Visualization



GARY ROCKSWOLD

THIRD EDITION

5.3

Exponential Functions and Models

- Distinguish between linear and exponential growth
- Model data with exponential functions
- Calculate compound interest
- Use the natural exponential functions in applications

Population Growth by a Constant Number vs by a Constant Percentage

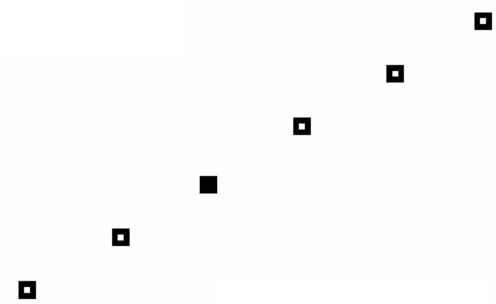
Suppose a population is 10,000 in January 2004. Suppose the population increases by…

- 500 people per year
- What is the population in Jan 2005?
  - 10,000 + 500 = 10,500
- What is the population in Jan 2006?
  - 10,500 + 500 = 11,000

- 5% per year
- What is the population in Jan 2005?
  - 10,000 + .05(10,000) = 10,000 + 500 = 10,500
- What is the population in Jan 2006?
  - 10,500 + .05(10,500) = 10,500 + 525 = 11,025

Suppose a population is 10,000 in Jan 2004. Suppose the population increases by 500 per year. What is the population in ….

- Jan 2005?
  - 10,000 + 500 = 10,500
- Jan 2006?
  - 10,000 + 2(500) = 11,000
- Jan 2007?
  - 10,000 + 3(500) = 11,500
- Jan 2008?
  - 10,000 + 4(500) = 12,000

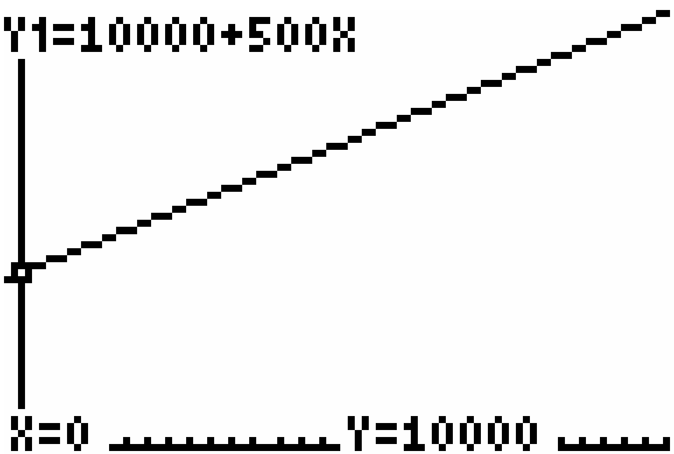


Suppose a population is 10,000 in Jan 2004 and increases by 500 per year.

- Let t be the number of years after 2004. Let P(t) be the population in year t. What is the symbolic representation for $P(t)$? We know…
- Population in 2004 = $P(0) = 10,000 + 0(500)$
- Population in 2005 = $P(1) = 10,000 + 1(500)$
- Population in 2006 = $P(2) = 10,000 + 2(500)$
- Population in 2007 = $P(3) = 10,000 + 3(500)$
- **Population t years after 2004 =**
  $P(t) = 10,000 + t(500)$

Population is 10,000 in 2004; increases by 500 per yr $P(t) = 10,000 + t(500)$

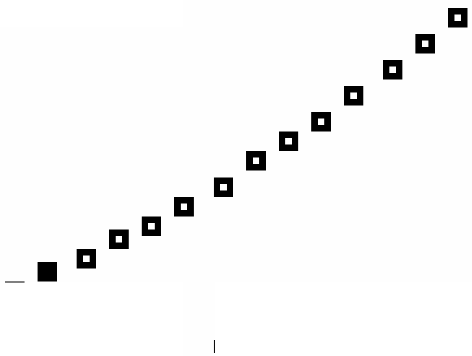
- *P* is a linear function of *t*.
- What is the slope?
  - 500 people/year
- What is the *y*-intercept?
  - number of people at time 0 (the year 2004) = 10,000



When *P* increases by a constant number of people per year, *P* is a linear function of *t*.

Suppose a population is 10,000 in Jan 2004. More realistically, suppose the population increases by **5% per year**.
What is the population in ….

- Jan 2005?
  - 10,000 + .05(10,000) = 10,000 + 500 = 10,500
- Jan 2006?
  - 10,500 + .05(10,500) = 10,500 + 525 = 11,025
- Jan 2007?
  - 11,025 + .05(11,025) = 11,025 + 551.25 = 11,576.25



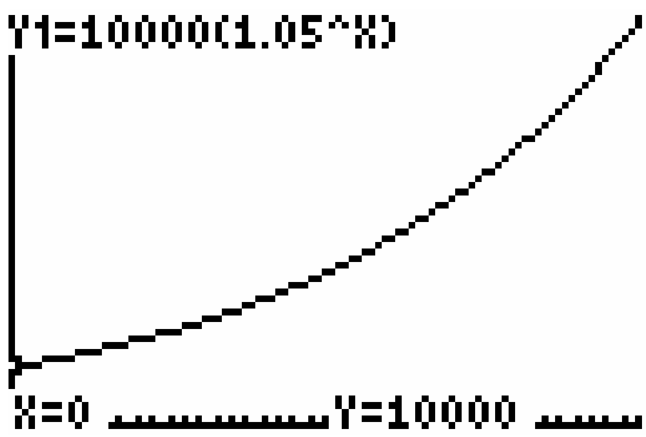
Suppose a population is 10,000 in Jan 2004 and increases by 5% per year.

- Let t be the number of years after 2004. Let $P(t)$ be the population in year t. What is the symbolic representation for $P(t)$? We know…
- Population in 2004 = $P(0)$ = 10,000
- Population in 2005 = $P(1)$ = 10,000 + .05 (10,000) = 1.05(10,000) = $1.05^1(10,000)$ =10,500
- Population in 2006 = $P(2)$ = 10,500 + .05 (10,500) = 1.05 (10,500) = 1.05 (1.05)(10,000) = $1.05^2(10,000)$ = 11,025
- **Population t years after 2004 =**

  $P(t) = 10,000(1.05)^t$

Population is 10,000 in 2004; increases by 5% per yr $P(t) = 10{,}000\,(1.05)^t$

- P is an EXPONENTIAL function of t. More specifically, an exponential growth function.
- What is the base of the exponential function?
  - 1.05
- What is the y-intercept?
  - number of people at time 0 (the year 2004) = 10,000



When P increases by a constant percentage per year, P is an exponential function of t.

Linear vs. Exponential Growth

- A Linear Function adds a fixed amount to the previous value of *y* for each unit increase in *x*
- For example, in $f(x) = 10{,}000 + 500x$ 500 is added to *y* for each increase of 1 in x.

- An Exponential Function multiplies a fixed amount to the previous value of *y* for each unit increase in *x*.
- For example, in $f(x) = 10{,}000\ (1.05)^x$ *y* is multiplied by 1.05 for each increase of 1 in *x*.

Definition of Exponential Function

- A function represented by

  $f(x) = Ca^x$, $a > 0$, a not 1, and $C > 0$

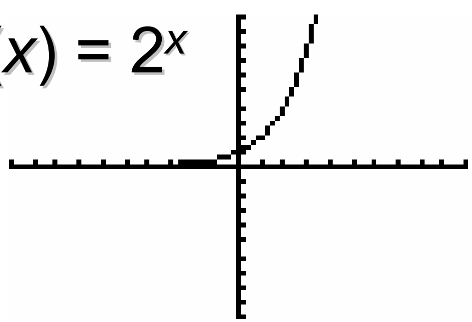
  is an exponential function with base a and coefficient C.

- If $a > 1$, then f is an exponential growth function
- If $0 < a < 1$, then f is an exponential decay function

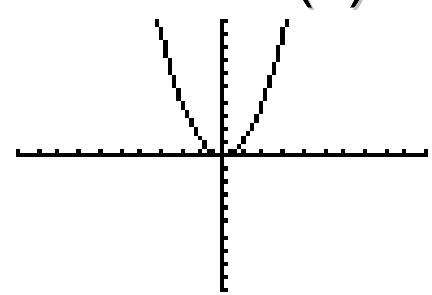
Caution

- Don't confuse $f(x) = 2^x$ with $f(x) = x^2$
- $f(x) = 2^x$ is an exponential function.
- $f(x) = x^2$ is a polynomial function, specifically a quadratic function.
- The functions and consequently their graphs are very different.

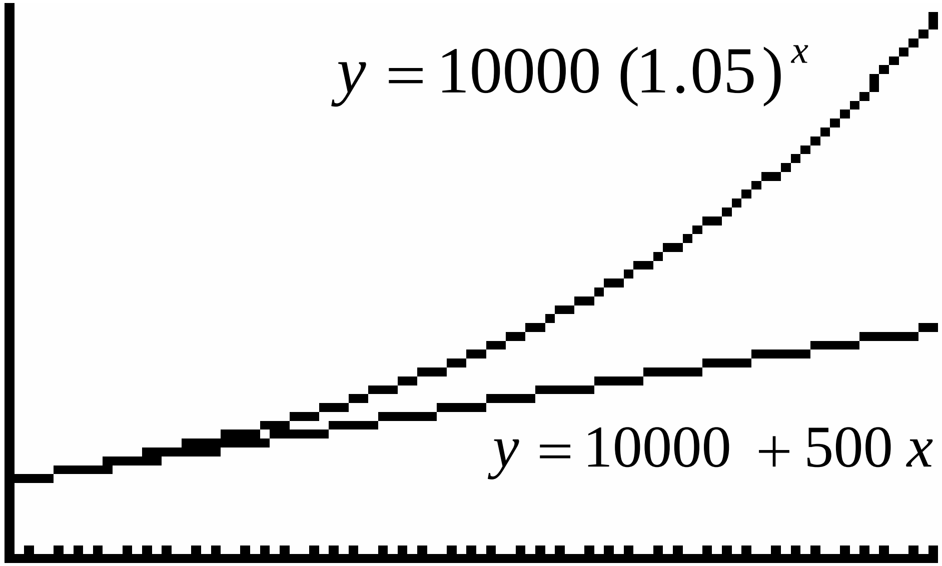
$f(x) = 2^x$

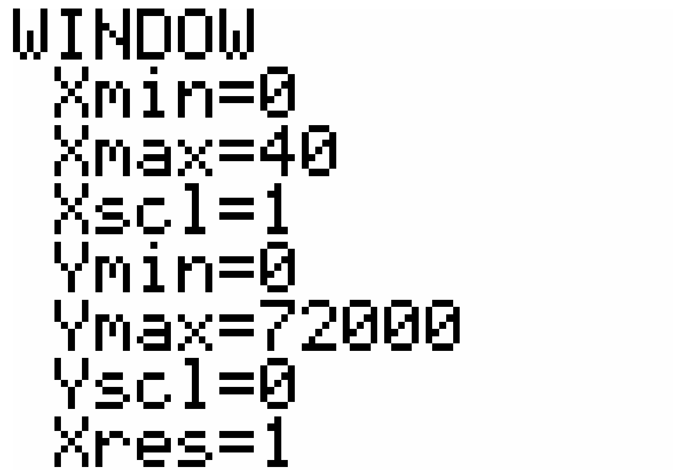


$f(x) = x^2$



Comparison of Exponential and Linear Functions





Linear Function

$y = 10000 + 500x$

x	y	?x	?y	$\frac{?y}{?x}$
0	10000			
1	10500	1	500	500/1=**500**
2	11000	1	500	500/1=**500**
3	11500	1	500	500/1=**500**
4	12000	1	500	500/1=**500**
5	12500	1	500	500/1=**500**
6	13000	1	500	500/1=**500**

Linear Function - Slope is constant.

Exponential Function

$Y = 10000\ (1.05)^x$

x	y	Ratios of consecutive y-values (corresponding to unit increases in x)
0	10,000	
1	10,500	10500/10000 = 1.05
2	11,025	11025/10500 = 1.05
3	11,576	11576/11025 = 1.05
4	12,155	12155/11576 = 1.05
5	12,763	12763/12155 = 1.05
6	13,401	13401/12763 = 1.05

Exponential Function - Ratios of consecutive y-values (corresponding to unit increases in x) are constant, in this case 1.05.

Note that this constant is the base of the exponential function.

Which function is linear and which is exponential?

x	*y*
-3	3/8
-2	3/4
-1	3/2
0	3
1	6
2	12
3	24

x	*y*
-3	9
-2	7
-1	5
0	3
1	1
2	-1
3	-3

For the linear function, tell the slope and *y*-intercept. For the exponential function, tell the base and the *y*-intercept. Write the equation of each.

Which function is linear and which is exponential? continued

x	y
-3	3/8
-2	3/4
-1	3/2
0	3
1	6
2	12
3	24

$(3/4)/(3/8) = 2$

$(3/2)/(3/4) = 2$

$3/(3/2) = 2$

$6/3 = 2$

$12/6 = 2$

$24/12 = 2$

y is an exponential function of *x* because the ratio of consecutive values of *y* is constant, namely 2.
Thus the base is 2.
The *y*-intercept is 3.
Thus the equation is
$y = 3 \cdot 2^x$

Which function is linear and which is exponential? continued

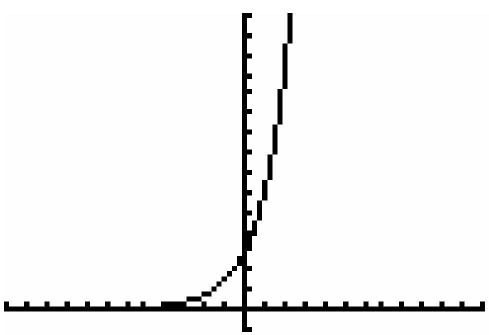
x	*y*
-3	9
-2	7
-1	5
0	3
1	1
2	-1
3	-3

y is a linear function of x because the slope is constant, namely $-2/1 = -2$. The y-intercept is 3. Thus the equation is $y = -2x + 3$

Exponential Growth vs Decay

- Example of exponential growth function

  $f(x) = 3 \cdot 2^x$

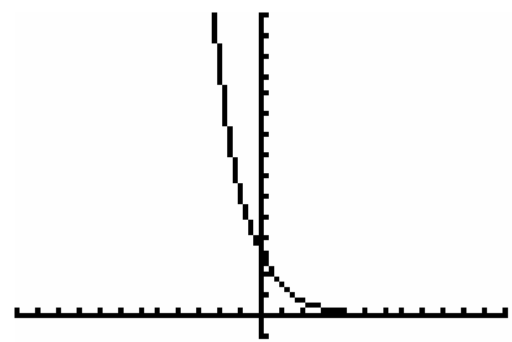


- Example of exponential decay function

$$f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$$

$$f(x) = 3\left(2^{-1}\right)^x$$

$$f(x) = 3 \cdot 2^{-x}$$



Recall

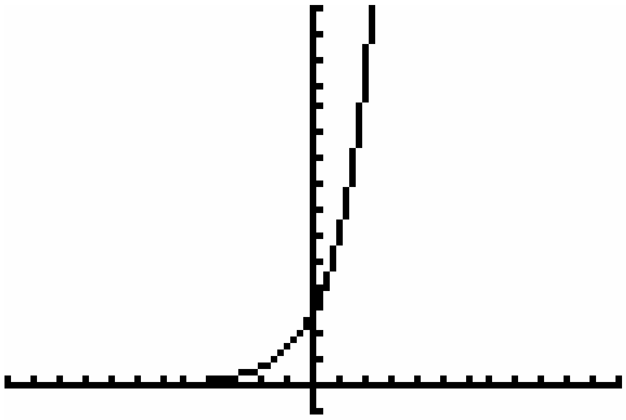
- In the exponential function

$$f(x) = Ca^x$$

  - If $a > 1$, then f is an exponential growth function
  - If $0 < a < 1$, then f is an exponential decay function

Exponential Growth Function $f(x) = Ca^x$ where $a > 1$

- Example
- $f(x) = 3 \cdot 2^x$



- Properties of an exponential growth function
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \infty)$
  - f increases on $(-\infty, \infty)$
  - The negative x-axis is a horizontal asymptote.
  - y-intercept is $(0,3)$.

Exponential Decay Function $f(x) = Ca^x$ where $0 < a < 1$

- Example

$$f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$$

$$f(x) = 3\left(2^{-1}\right)^x$$

$$f(x) = 3 \cdot 2^{-x}$$

- Properties of an exponential decay function
  - Domain: (-∞, ∞)
  - Range: (0, ∞)
  - *f* decreases on (-∞, ∞)
  - The positive *x*-axis is a horizontal asymptote.
  - *y*-intercept is (0,3).

Example of exponential decay - Carbon-14 dating

- The time it takes for half of the atoms to decay into a different element is called the half-life of an element undergoing radioactive decay.
- The half-life of carbon-14 is 5700 years.
- Suppose C grams of carbon-14 are present at $t = 0$. Then after 5700 years there will be $C/2$ grams present.

Recall the half-life of carbon-14 is 5700 years.

- Let t be the number of years.
- Let $A = f(t)$ be the amount of carbon-14 present at time t.
- Let C be the amount of carbon-14 present at $t = 0$.
- Then $f(0) = C$ and $f(5700) = C/2$.
- Thus two points of f are $(0, C)$ and $(5700, C/2)$
- Using the point $(5700, C/2)$ and substituting 5700 for t and $C/2$ for A in $A = f(t) = Ca^t$ *yields:*

$$C/2 = C\,a^{5700}$$

- Dividing both sides by C yields: $1/2 = a^{5700}$

Recall the half-life of carbon-14 is 5700 years.

$$\frac{1}{2} = a^{5700}$$

Raising both sides to the 1/5700 power gives

$$\left(\frac{1}{2}\right)^{\frac{1}{5700}} = a$$

So $A = f(t) = Ca^t$ becomes

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{1}{5700}}\right]^t$$

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{t}{5700}}\right]$$

Half-life

Generalizing this

- If a radioactive sample containing *C* units has a half-life of *k* years, then the amount *A* remaining after *x* years is given by

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Example of Radioactive Decay

- Radioactive strontium-90 has a half-life of about 28 years and sometimes contaminates the soil. After 50 years, what percentage of a sample of radioactive strontium would remain?

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Note calculator keystrokes:

```
.5^(50/28)
        .2900323465
```

$$A(50) = C\left(\frac{1}{2}\right)^{\frac{50}{28}} \approx C(.2900323465)$$

Since C is present initially and after 50 years .29C remains, then 29% remains.

Example of Exponential Growth - Compound Interest

- Suppose $10,000 is deposited into an account which pays 5% interest compounded annually. Then the amount *A* in the account after *t* years is:

$$A(t) = 10{,}000\,(1.05)^t$$

- Note the similarity with: Suppose a population is 10,000 in 2004 and increases by 5% per year. Then the population *P*, *t* years after 2004 is:

$$P(t) = 10{,}000\,(1.05)^t$$

Frequencies of Compounding (Adding Interest)

- annually (1 time per year)
- semiannually (2 times per year)
- quarterly (4 times per year)
- monthly (12 times per year)
- daily (365 times per year)

Compound Interest Formula

- If P dollars is deposited in an account paying an annual rate of interest r, compounded (paid) n times per year, then after t years the account will contain A dollars, where

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Suppose $1000 is deposited into an account yielding 5% interest compounded at the following frequencies. How much money after *t* years?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Annually $\quad A = 1000\left(1+\frac{.05}{1}\right)^{1t} = 1000(1.05)^{t}$
- Semiannually $\quad A = 1000\left(1+\frac{.05}{2}\right)^{2t} = 1000(1.025)^{2t}$
- Quarterly $\quad A = 1000\left(1+\frac{.05}{4}\right)^{4t} = 1000(1.0125)^{4t}$
- Monthly $\quad A = 1000\left(1+\frac{.05}{12}\right)^{12t} = 1000(1.0041\overline{6})^{12t}$

The Natural Exponential Function

- The function *f*, represented by

$$f(x) = e^x$$

is the natural exponential function where

$e \approx 2.718281828$

Continuously Compounded Interest

- If a principal of P dollars is deposited in an account paying an annual rate of interest r (expressed in decimal form), compounded continuously, then after t years the account will contain A dollars, where

$$A = Pe^{rt}$$

Example

- Suppose $100 is invested in an account with an interest rate of 8% compounded continuously. How much money will there be in the account after 15 years?

$$A = Pe^{rt}$$

$$A = \$100\, e^{.08(15)}$$

$$A = \$332.01$$

5.4

Logarithmic Functions and Models

- Evaluate the common logarithm function
- Solve basic exponential and logarithmic equations
- Evaluate logarithms with other bases
- Solve general exponential and logarithmic equations

Common Logarithm

- The common logarithm of a positive number x, denoted $\log x$, is defined by

  $$\log x = k \text{ if and only if } x = 10^k$$

  where k is a real number.

- The function given by $f(x) = \log x$ is called the common logarithm function.

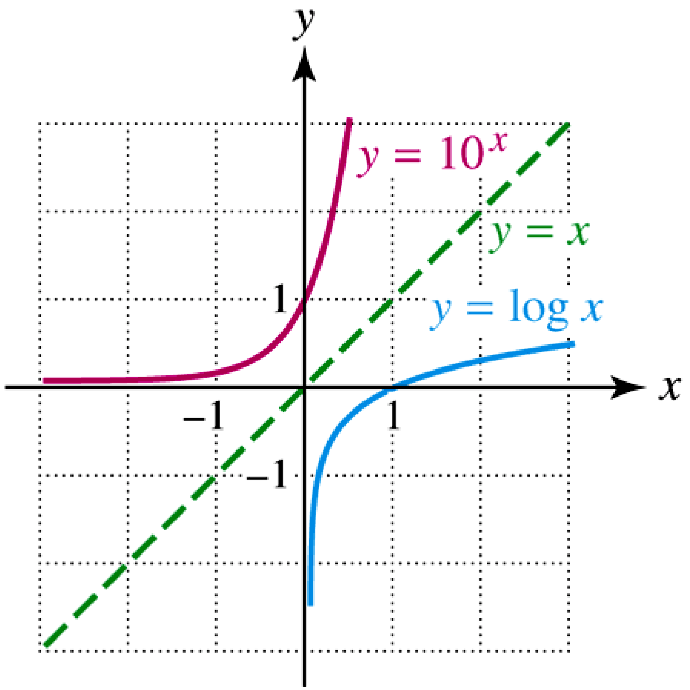
Evaluate each of the following.

- log10
- log 100
- log 1000
- log 10000
- log (1/10)
- log (1/100)
- log (1/1000)
- log 1

- **1** because $10^1 = 10$
- **2** because $10^2 = 100$
- **3** because $10^3 = 1000$
- **4** because $10^4 = 10000$
- **–1** because $10^{-1} = 1/10$
- **–2** because $10^{-2} = 1/100$
- **–3** because $10^{-3} = 1/1000$
- **0** because $10^0 = 1$

Graph of $f(x) = \log x$

x	$f(x)$
.01	-2
.1	-1
1	0
10	1
100	2



Note that the graph of $y = \log x$ is the graph of $y = 10^x$ reflected through the line $y = x$. This suggests that these are inverse functions.

The Inverse of $y = \log x$

- Note that the graph of $f(x) = \log x$ passes the horizontal line test so it is a 1-1 function and has an inverse function.
- Find the inverse of $y = \log x$
- Using the definition of common logarithm to solve for x gives
- $x = 10^y$
- Interchanging x and y gives
- $y = 10^x$
- So yes, the inverse of $y = \log x$ is $y = 10^x$

Inverse Properties of the Common Logarithm

- Recall that $f^{-1}(x) = 10^x$ given $f(x) = \log x$
- Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}
  - **$\log(10^x) = x$ for all real numbers x.**
- Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f
  - **$10^{\log x} = x$ for any positive number x**

Solving Exponential Equations Using The Inverse Property $\log(10^x) = x$

- Solve the equation $10^x = 35$
- Take the common log of both sides
  - $\log 10^x = \log 35$
- Using the inverse property $\log(10^x) = x$ this simplifies to
  - $x = \log 35$
- Using the calculator to estimate log 35 we have
  - $x \approx 1.54$

Solving Logarithmic Equations Using The Inverse Property $10^{\log x} = x$

- Solve the equation $\log x = 4.2$
- Exponentiate each side using base 10
  - $10^{\log x} = 10^{4.2}$
- Using the inverse property $10^{\log x} = x$ this simplifies to
  - $x = 10^{4.2}$
- Using the calculator to estimate $10^{4.2}$ we have
  - $x \approx 15848.93$

Definition of Logarithm With Base *a*

- The logarithm with base a of a positive number x, denoted by $\log_a x$ is defined by
  $\log_a x = k$ if and only if $x = a^k$
  where $a > 0$, $a \neq 1$, and k is a real number.

- The function given by $f(x) = \log_a x$ is called the logarithmic function with base a.

Practice with the Definition

Practice Questions:

- $\text{Log}_b c = d$ means____
- $p = \log_y m$ means ____
- True or false:
  - True or false: $\log_2 8 = 3$
  - True or false: $\log_5 25 = 2$
  - True or false: $\log_{25} 5 = 1/2$
  - True or false: $\log_4 8 = 2$

Answers:

- $b^d = c$
- $y^p = m$
- True because $2^3=8$
- True because $5^2=25$
- True because $25^{1/2}=5$
- False because $4^2=16$ not 8

What is the value of $\log_4 8$? It is 3/2 because $4^{3/2} = 8$

Practice Evaluating Logarithms

Evaluate

- $\log_6 36$
- $\log_{36} 6$
- $\log_2 32$
- $\log_{32} 2$
- $\log_6 (1/36)$
- $\log_2 (1/32)$
- $\log 100$
- $\log (1/10)$
- $\log 1$

Answers:

- **2** because $6^2 = 36$
- **1/2** because $36^{(1/2)} = 6$
- **5** because $2^5 = 32$
- **1/5** because $32^{(1/5)} = 2$
- **–2** because $6^{-2} = 1/36$
- **–5** because $2^{-5} = 1/32$
- **2** because $10^2 = 100$
- **–1** because $10^{-1} = 1/10$
- **0** because $10^0 = 1$

Calculators and logarithms

- The TI-83 evaluates base 10 logarithms and base e logarithms.
- Base 10 logs are called common logs.
  - log *x* means $\log_{10}x$.
  - Notice the log button on the calculator.
- Base e logs are called natural logs.
  - ln *x* means $\log_e x$.
  - Notice the ln button on the calculator.

Evaluate each of the following without calculator. Then check with calculator.

- $\ln e$
- $\ln(e^2)$
- $\ln 1$
- $\ln\sqrt{e}$

- $\ln e = \log_e e = 1$ since $e^1 = e$
- $\ln(e^2) = \log_e(e^2) = 2$ since 2 is the exponent that goes on e to produce e^2.
- $\ln 1 = \log_e 1 = 0$ since $e^0 = 1$
- 1/2 since 1/2 is the exponent that goes on e to produce $e^{1/2}$

The Inverse of $y = \log_a x$

- Note that the graph of $f(x) = \log_a x$ passes the horizontal line test so it is a 1-1 function and has an inverse function.
- Find the inverse of $y = \log_a x$
- Using the definition of common logarithm to solve for x gives
- $x = a^y$
- Interchanging x and y gives
- $y = a^x$
- So the inverse of $y = \log_a x$ is $y = a^x$

Inverse Properties of Logarithms With Base a

- Recall that $f^{-1}(x) = a^x$ given $f(x) = \log_a x$
- Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}
  - **$\log_a(a^x) = x$ for all real numbers x.**
- Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f
  - **$a^{\log_a x} = x$ for any positive number x**

Solving Exponential Equations Using The Inverse Property $\log_a(a^x) = x$

- Solve the equation $4^x = 1/64$
- Take the log of both sides to the base 4
  - $\log_4 (4^x) = \log_4(1/64)$
- Using the inverse property $\log_a (a^x) = x$ this simplifies to
  - $x = \log_4(1/64)$
- Since 1/64 can be rewritten as 4^{-3}
  - $x = \log_4(4^{-3})$
- Using the inverse property $\log_a (a^x) = x$ this simplifies to
  - $x = -3$

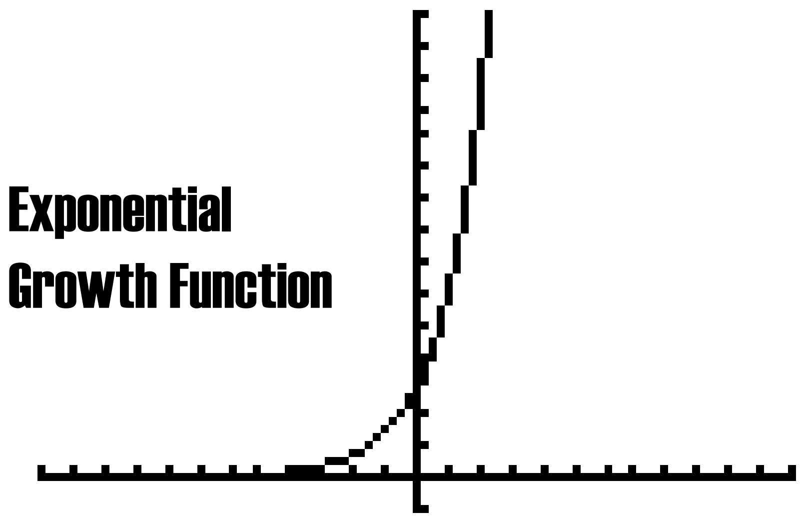
Solving Exponential Equations Using The Inverse Property $\log_a(a^x) = x$

- Solve the equation $e^x = 15$
- Take the log of both sides to the base e
  - $\ln(e^x) = \ln(15)$
- Using the inverse property $\log_a(a^x) = x$ this simplifies to
  - $x = \ln 15$
- Using the calculator to estimate ln 15
  - $x \approx 2.71$

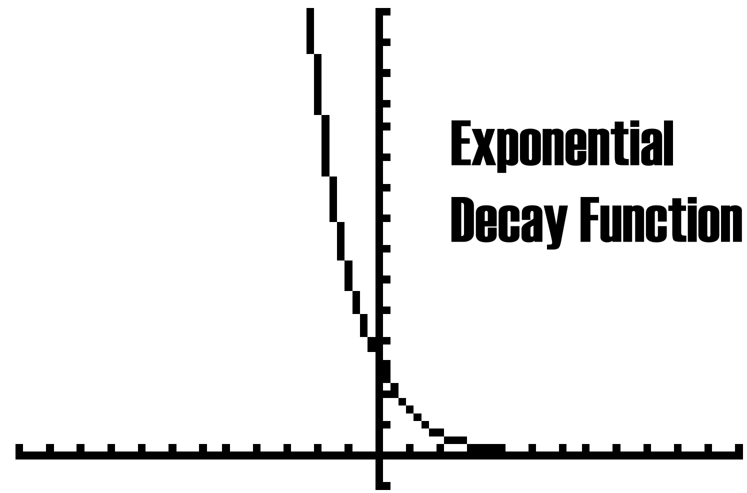
Solving Logarithmic Equations Using The Inverse Property $a^{\log_a x} = x$

- Solve the equation $\ln x = 1.5$
- Exponentiate both sides using base *e*
  - $e^{\ln x} = e^{1.5}$
- Using the inverse property $a^{\log_a x} = x$ this simplifies to
  - $x = e^{1.5}$
- Using the calculator to estimate $e^{1.5}$
  - $x \approx 4.48$

Recall from section 5.3

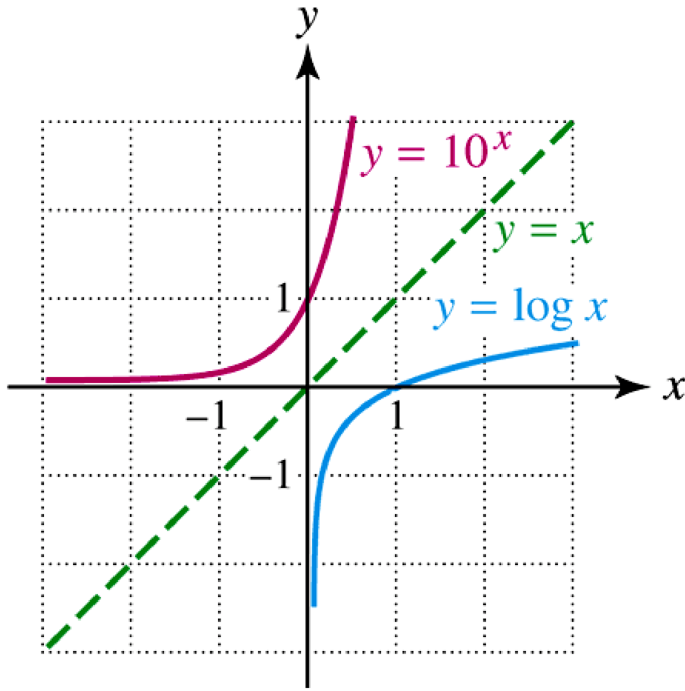


Graph of $f(x) = a^x$ where $a > 1$

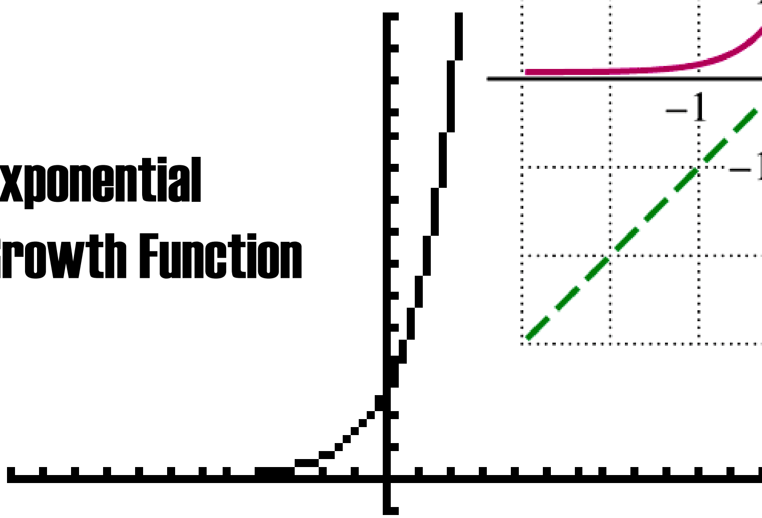


Graph of $f(x) = a^x$ where $0 < a < 1$

Using the fact that the graph of a function and its inverse are symmetric with respect to the line $y = x$, **graph $f^{-1}(x) = \log_a x$** for the two types of exponential functions listed above. Looking at the two resulting graphs, what is the domain of a logarithmic function? What is the range of a logarithmic function?

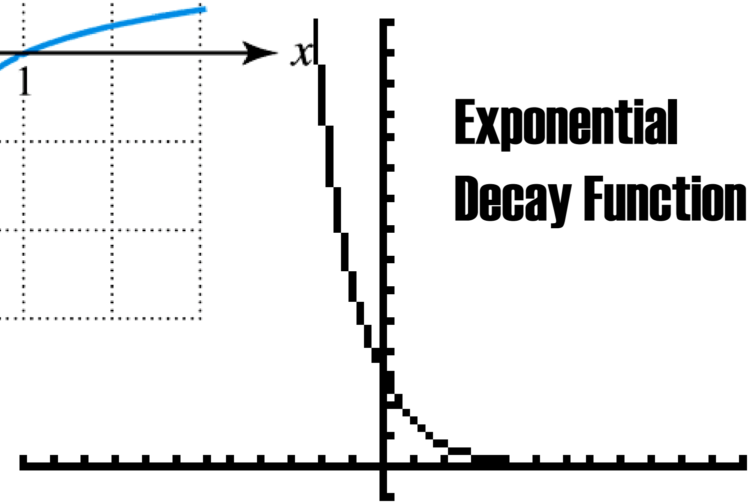


Exponential Growth Function



Graph of $f(x) = a^x$ where $a > 1$

Exponential Decay Function



Graph of $f(x) = a^x$ where $0 < a < 1$

Superimpose graphs of the inverses of the functions above similar to Figure 5.58 on page 422