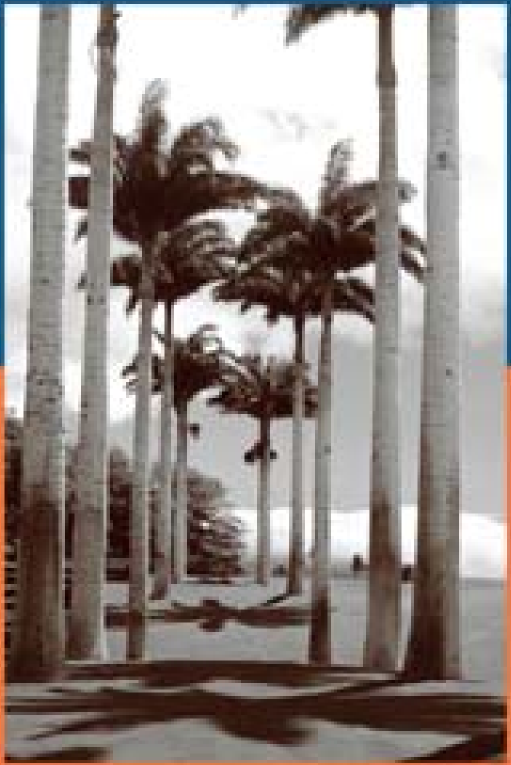
COLLEGE ALGEBRA

with Modeling and Visualization



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THIRD EDITION

8.1

Sequences

- Understand basic concepts about sequences
- Learn how to represent sequences
- Identify and use arithmetic sequences
- Identify and use geometric sequences

Introduction

- A sequence is a function that computes an ordered list.

Example

If an employee earns $12 per hour, the function $f(n) = 12n$ generates the terms of the sequence 12, 24, 36, 48, 60, …

when $n = 1, 2, 3, 4, 5, \ldots$

SEQUENCE

An **infinite sequence** is a function that has the set of natural numbers as its domain. A **finite sequence** is a function with domain $D = \{1, 2, 3, \ldots, n\}$, for some fixed natural number n.

Sequences

- Instead of letting *y* represent the output, it is common to write $a_n = f(n)$, where *n* is a natural number in the domain of the sequence. The *terms* of a sequence are

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

    - The first term is $a_1 = f(1)$, the second term is $a_2 = f(2)$ and so on. The *nth term* or *general term* of a sequence is $a_n = f(n)$.

Example

Write the first four terms a_1, a_2 , a_3, a_4,… of each sequence, where $a_n = f(n)$,

a) $f(n) = 5n + 3$

b) $f(n) = (4)^{n-1} + 2$

Solution

a) $a_1 = f(1) = 5(1) + 3 = 8$

$a_2 = f(2) = 5(2) + 3 = 13$

$a_3 = f(3) = 5(3) + 3 = 18$

$a_4 = f(4) = 5(4) + 3 = 23$

b) $a_1 = f(1) = (4)^{1-1} + 2 = 2$

$a_2 = f(2) = (4)^{2-1} + 2 = 6$

$a_3 = f(3) = (4)^{3-1} + 2 = 18$

$a_4 = f(4) = (4)^{4-1} + 2 = 66$

Recursive Sequence

- With a recursive sequence, one or more previous terms are used to generate the next term.
- The terms a_1 through a_{n-1} must be found before a_n can be found.

Example

a) Find the first four terms of the recursive sequence that is defined by

$a_n = 3a_{n-1} + 5$ and $a_1 = 4$, where $n \geq 2$.

b) Graph the first 4 terms of the sequence.

Example continued

Solution

a) Numerical Representation

$$a_1 = 4$$

$$a_2 = 3a_1 + 5 = 3(4) + 5 = 17$$

$$a_3 = 3a_2 + 5 = 3(17) + 5 = 56$$

$$a_4 = 3a_3 + 5 = 3(56) + 5 = 173$$

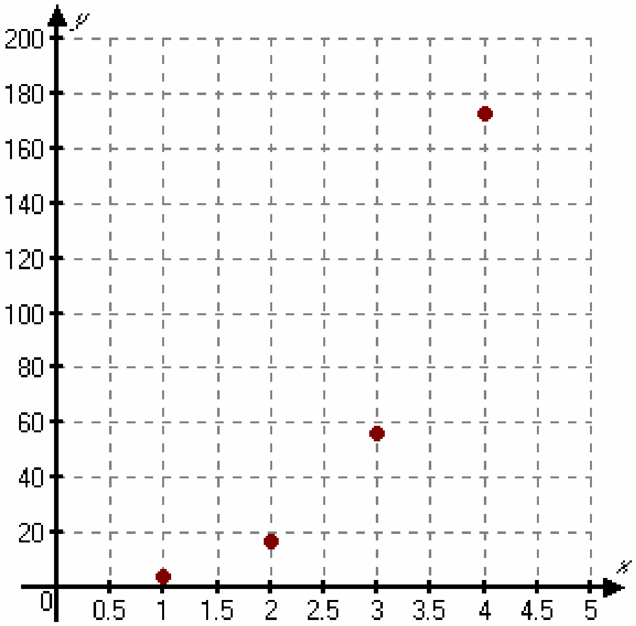
The first four terms are 4, 17, 56, and 173.

n	1	2	3	4
a_n	4	17	56	173

Solution continued

Graphical Representation

b) To represent these terms graphically, plot the points. Since the domain of the graph only contains natural numbers, the graph of the sequence is a scatterplot.



Arithmetic Sequences

INFINITE ARITHMETIC SEQUENCE

An **infinite arithmetic sequence** is a linear function whose domain is the set of natural numbers.

Example

An employee receives 10 vacations days per year. Thereafter the employee receives an additional 2 days per year with the company. The amount of vacation days after n years with the company is represented by

$f(n) = 2n + 10$, where f is a linear function.

How many vacation days does the employee have after 14 years? (Assume no rollover of days.)

Solution $f(14) = 2(14) + 10 = 38$ days of vacation.

Arithmetic Sequence

- An arithmetic sequence can be defined recursively by $a_n = a_{n-1} + d$, where d is a constant. Since $d = a_n - a_{n-1}$ for each valid n, d is called the *common difference*. If $d = 0$, then the sequence is a *constant sequence*. A *finite arithmetic sequence* is similar to an infinite arithmetic sequence except its domain is $D = \{1, 2, 3, \ldots, n)$, where n is a fixed natural number.
- Since an arithmetic sequence is a linear function, it can always be represented by $f(n) = dn + c$, where d is the common difference and c is a constant.

Example

Find a general term $a_n = f(n)$ for the arithmetic sequence; $a_1 = 4$ and $d = –3$.

Solution

Let $f(n) = dn + c$.

Since $d = –3$, $f(n) = –3n + c$.

$$a_1 = f(1) = -3(1) + c = 4 \quad \text{or} \quad c = 7$$

Thus $a_n = –3n + 7$.

Terms of an Arithmetic Sequence

nth TERM OF AN ARITHMETIC SEQUENCE

In an arithmetic sequence with first term a_1 and common difference d, the nth term, a_n, is given by

$$a_n = a_1 + (n - 1)d.$$

Example

Find a symbolic representation (formula) for the arithmetic sequence given by 6, 10, 14, 18, 22,…

Solution

The first term is 6. Successive terms can be found by adding 4 to the previous term. $a_1 = 6$ and $d = 4$

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ &= 6 + (n - 1)(4) \\ &= 4n + 2 \end{aligned}$$

Geometric Sequences

- Geometric sequences are capable of either rapid growth or decay.

Examples

- Population
- Salary
- Automobile depreciation

INFINITE GEOMETRIC SEQUENCE

An **infinite geometric sequence** is a function defined by $f(n) = cr^{n-1}$, where c and r are nonzero constants. The domain of f is the set of natural numbers.

Example

Find a general term a_n for the geometric sequence; a_3 = 18 and a_6 = 486.

Solution

Find $a_n = cr^{n-1}$ so that $a_3 = 18$ and $a_6 = 486$.

Since $\dfrac{a_6}{a_3} = \dfrac{cr^{6-1}}{cr^{3-1}} = \dfrac{r^5}{r^2} = r^3$ and $\dfrac{a_6}{a_3} = \dfrac{486}{18} = 27,$

$$r^3 = 27 \text{ or } r = 3.$$

So $a_n = c(3)^{n-1}$.

Therefore $a_3 = c(3)^{3-1} = 18$ or $c = 2$.

Thus $a_n = 2(3)^{n-1}$.

8.2

Series

- Understand basic concepts about series
- Identify and find the sum of arithmetic series
- Identify and find the sum of geometric series
- Learn and use summation notation

Introduction

- A *series* is the summation of the terms in a sequence.
- *Series* are used to approximate functions that are too complicated to have a simple formula.
- *Series* are instrumental in calculating approximations of numbers like π and e.

SERIES

A **finite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n,$$

and an **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

Infinite Series

- An infinite series contains many terms.
- Sequence of partial sums:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

- If S_n approaches a real number S as $n \to \infty$, then the sum of the infinite series is S.
- Some infinite series do not have a sum S.

Example

For $a_n = 3n - 1$, calculate S_5.

Solution

Since $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$, start calculating the first five terms of the sequence $a_n = 3n - 1$.

$a_1 = 3(1) - 1 = 2$

$a_2 = 3(2) - 1 = 5$

$a_3 = 3(3) - 1 = 8$

$a_4 = 3(4) - 1 = 11$

$a_5 = 3(5) - 1 = 14$

Thus $S_5 = 2 + 5 + 8 + 11 + 14 = 40$

Arithmetic Series

- Summing the terms of a arithmetic sequence results in an *arithmetic series*.

SUM OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE

The **sum of the first n terms of an arithmetic sequence**, denoted S_n, is found by averaging the first and nth terms and then multiplying by n. That is,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = n\left(\frac{a_1 + a_n}{2}\right).$$

Arithmetic Series continued

Since $a_n = a_1 + (n - 1)d$, S_n can also be written in the following way.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$$= \frac{n}{2}\left(a_1 + a_1 + (n-1)d\right)$$

$$= \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

Example

Use the formula to find the sum of the arithmetic series $4+7+10+\cdots+58$.

Solution

The series has n = 19 terms with a_1 = 4 and a_{19} = 58. We can then use the formula to find the sum.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$$S_{19} = 19\left(\frac{4+58}{2}\right)$$

$$= 589$$

Example

A worker has a starting annual salary of \$45,000 and receives a \$2500 raise each year. Calculate the total amount earned over 5 years.

Solution

The arithmetic sequence describing the salary during year *n* is computed by

$$a_n = 45{,}000 + 2500(n - 1).$$

The first and fifth year's salaries are

$$a_1 = 45{,}000 + 2500(1 - 1) = 45{,}000$$

$$a_5 = 45{,}000 + 2500(5 - 1) = 55{,}000$$

Solution continued

Thus the total amount earned during this 5-year period is

$$S_5 = 5\left(\frac{45,000 + 55,000}{2}\right) = \$250,000.$$

The sum can also be found using

$$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right).$$

$$S_5 = \frac{5}{2}\left(2 \cdot 45,000 + (5-1)2500\right) = \$250,000.$$

Geometric Series

- The sum of the terms of a geometric sequence is called a *geometric series*.

SUM OF THE FIRST *n* TERMS OF A GEOMETRIC SEQUENCE

If a geometric sequence has first term a_1 and common ratio r, then the sum of the first n terms is given by

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right),$$

provided $r \neq 1$.

Annuities

- An *annuity* is a sequence of deposits made at equal periods of time.
- After n years the amount is given by

$$A_0 + A_0(1+i) + A_0(1+i)^2 + \cdots + A_0(1+i)^{n-1}.$$

- This is a geometric series with first term $a_1 = A_0$ and common ratio $r = (1 + i)$. The sum of the first n terms is given by

$$S_n = A_0\left(\frac{1-(1+i)^n}{1-(1+i)}\right) = A_0\left(\frac{(1+i)^n - 1}{i}\right).$$

Example

A 30-year-old employee deposits $4000 into an account at the end of each year until age 65. If the interest rate is 8%, find the future value of the annuity.

Solution Let A_0 = 4000, i = 0.08, and n = 35. The future value of the annuity is given by

$$S_n = A_0\left(\frac{(1+i)^n - 1}{i}\right)$$

$$= 4000\left(\frac{(1+0.08)^{35} - 1}{0.08}\right)$$

$$= \$689,267.$$

Infinite Geometric Series

SUM OF AN INFINITE GEOMETRIC SEQUENCE

The sum of the infinite geometric sequence with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1 - r},$$

provided $|r| < 1$. If $|r| \geq 1$, then this sum does not exist.

Summation Notation

- *Summation notation* is used to write series efficiently. The symbol $\sum$, sigma, indicates the sum.

SUMMATION NOTATION

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

- The letter *k* is called the *index of summation*. The numbers 1 and *n* represent the subscripts of the first and last term in the series. They are called the *lower limit* and *upper limit* of the summation, respectively.

Example

Evaluate each series.

a) $\sum_{k=1}^{3} 4k$ b) $\sum_{k=1}^{3} 4$ c) $\sum_{k=1}^{6} (3k+6)$

Solution

a) $\sum_{k=1}^{3} 4k = 4+8+12 = 24$ b) $\sum_{k=1}^{3} 4 = 4+4+4 = 12$

c) $$\sum_{k=1}^{6} (3k+6) = \big(3(1)+6\big)+\big(3(2)+6\big)+\big(3(3)+6\big)+$$

$$\big(3(4)+6\big)+\big(3(5)+6\big)+\big(3(6)+6\big)$$

$$= 9+12+15+18+21+24 = 99$$

Properties for Summation Notation

PROPERTIES FOR SUMMATION NOTATION

Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ be sequences, and c be a constant.

1. $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
2. $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
3. $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
4. $\displaystyle\sum_{k=1}^{n} c = nc$
5. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
6. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Example

Use properties for summation notation to find each sum.

a) $\sum_{k=1}^{18} 4k$ b) $\sum_{k=1}^{11} 4k^2 - 6$

Solution

a) $\sum_{k=1}^{18} 4k = 4\sum_{k=1}^{18} k$

$= 4 \cdot \frac{18(18+1)}{2}$

$= 684$

b) $\sum_{k=1}^{11} 4k^2 - 6 = \sum_{k=1}^{11} 4k^2 - \sum_{k=1}^{11} 6$

$= 4\sum_{k=1}^{11} k^2 - \sum_{k=1}^{11} 6$

$= 4 \cdot \frac{11(11+1)(2 \cdot 11+1)}{6} - 11(6)$

$= 1958$