How do you raise a number to an irrational power? For example, 3^2 is just 3•3 = 9. But how do you compute $3^{\sqrt{2}}$ – without a calculator?

In algebra textbooks, you are often told to take on faith that your calculator knows how to compute expressions like $3^{\sqrt{2}}$. No explanation is given on how to raise a number to an irrational power without a calculator. Fortunately, some relatively easy calculus can de-mystify the process of computing a number to an irrational power.

The calculus explanation of computing a number to a power starts with an apparently un-related topic: computing the anti-derivative of the reciprocal function.

First, note that the reciprocal function $f(t) = \frac{1}{t}$ is continuous from 1 to ∞.

Continuity guarantees that an anti-derivative exists. Unfortunately, the anti-derivative cannot be expressed using familiar functions like t^2 or t^3. So we have to define a new type of function, called the logarithm function *ln(x)*, as the anti-derivative:

$$(1)\ \ln(x) = \int_1^x \frac{1}{t}\,dt$$

Immediate application of the Fundamental Theorem of Calculus gives us this useful result:

$$(2)\ \frac{d\ln(x)}{dx} = \frac{1}{x}$$

Next we will define the special number *e* as:

$$(3)\ \ln(e) = \int_1^e \frac{1}{t}\,dt = 1$$

Using the derivative formula in (2), you can quickly verify the Power Rule of Logarithms:

$$(4)\ \ln(x^r) = r\ln(x)$$

Because *ln(x)* is 1-to-1, this function has an inverse. Applying the Power Rule of Logarithms, you can verify that the inverse for *ln(x)* is e^x :

$$(5)\ \ln(e^x) = x \quad \text{and} \quad \mathrm{e}^{\ln(x)} = x$$

Because the range of *ln(x)* is $(-\infty,\infty)$, the domain of its inverse e^x must be $(-\infty,\infty)$. This last result is great news; it means we may input any real number, including irrational numbers, as powers!

But knowing that e^x exists and must equal some real number is just half the battle. How does one proceed to actually compute e^x for any real number?

To compute e^x, one uses a Taylor series expansion for e^x.

Also, our original question asked about computing powers of 3, namely $3^{\sqrt{2}}$. Do the results above also apply to exponential functions of other bases? Yes, because you can always convert from any base to base *e* without any loss of meaning.

For example, just convert 3^x to the base *e* through change-of-base formulas:

$$(6)\quad f(x) = 3^x \Leftrightarrow f(x) = e^{(\ln 3)x}$$

In other words, any exponential function can be converted to any desired base, and the values of that exponential function can be approximated through use of a Taylor series expansion.