This problem comes from **MATH 2412 Precalculus** – David Katz

Is there an easier way to do the algebra to transform a conic equation to eliminate the *xy*-term?

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The short answer is yes! Using matrices to represent the conic equation and the appropriate rotation, we can transform the equation quickly into one with no *xy*-term. (The following discussion was adapted from a book on matrices by A. Pettofrezzo).

First, compare the general form for a conic equation in $x$ and $y$ with coefficients $A,B,C,D,E,F$ with the corresponding matrix notation for the same conic equation:

(1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \leftrightarrow \quad \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

The next step is to determine a matrix to represent the rotation of the axes about the origin that transforms the conic equation such that the $B$-term vanishes, and the conic's axes are now parallel or perpendicular with the coordinate axes. We will let $(x, y)$ represent a point before the rotation, and let $(x', y')$ represent coordinates of the same point after the rotation relative to the new axes in $x'$ and $y'$. The angle for this rotation is computed from the formula $\cot(2\theta) = \frac{A-C}{B}$.

We can now set up the matrix multiplication which rotates the axes by $\theta$ degrees about the origin:

(2) $$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}$$

Extending this matrix arithmetic to three dimensions, we have an equivalent representation for this rotation:

(3) $$\begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y & 1 \end{pmatrix}$$

The extra dimension will be necessary so that we can multiply the rotation matrix with the other three-dimensional matrices in statement (1). In a moment, we will also need the transpose of statement (3). Recall that the transpose of a matrix product equals the product of the transposes in reverse order:

(4) $$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Finally substituting statements (3) and (4) into statement (1), we have:

$$(5)\qquad \begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0$$

Well there it is! The above matrix multiplication rotates the axes such that the conic equation in statement (1) no longer has an $Bxy$ term, and the conic's axes are now parallel or perpendicular to the $x'$ and $y'$ axes. The big advantage here is that the 3x3 matrix multiplication in statement (5) can be done quickly on a calculator or spreadsheet.

## An Example of This Matrix Method (adapted from Stewart's Calculus textbook)

Rotate the conic equation $73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0$ to eliminate the $72xy$ term. Determining the angle of rotation gives these values for sine and cosine:

$$\cot(2\theta) = \frac{73-52}{72} = \frac{7}{24} \to \cos\theta = \frac{4}{5} \text{ and } \sin\theta = \frac{3}{5}$$

Thus, the matrix multiplication we need is:

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 73 & \frac{72}{2} & \frac{30}{2} \\ \frac{72}{2} & 52 & -\frac{40}{2} \\ \frac{30}{2} & -\frac{40}{2} & -75 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0$$

Completing the matrix multiplication in the middle, we get:

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{pmatrix} 100 & 0 & 0 \\ 0 & 25 & -25 \\ 0 & -25 & -75 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0$$

Writing this remaining matrix multiplication as a single equation, we get:

$$100(x')^2 + 25(y')^2 - 50y' - 75 = 0$$

Simplifying by factoring out 25 and replacing $x'$ and $y'$ with just $x$ and $y$, we get:

$$4x^2 + y^2 - 2y - 3 = 0$$

Observe that the conic equation represents an ellipse with center at (0,1).