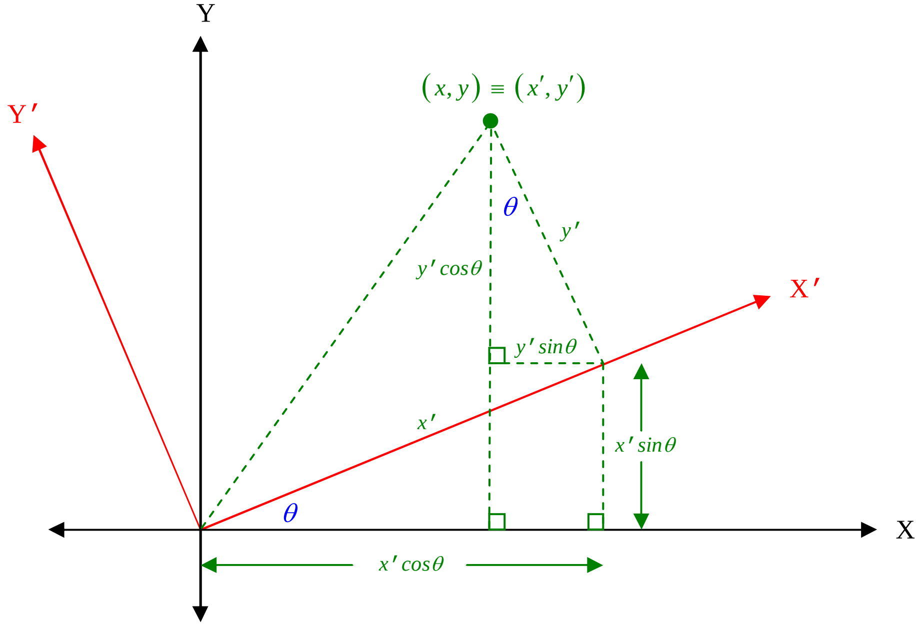
Conic Sections - Axis Rotation

The equation (1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents a rotated conic section in the xy-plane or a degenerate thereof, where $A - F$ are real constants and B is non-zero; of course, if the coefficient B is identical to zero, then the conic is considered "unrotated" and thus its axis is either horizontal or vertical and can be analyzed completely as in previous discussions.

It remains now to be shown that the above equation can be transformed into an unrotated conic in the new $x'y'$ coordinate system upon calculating this rotation angle "θ", yielding the newly-transformed equation (2) $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$, where for proper choice of θ, B' will equal zero.

Consider the figure:



The following transformation equations are evident from the above figure:

(3) $\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$ or in matrix form: $B[\mathbf{x}]_B = B'[\mathbf{x}]_{B'} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$.

Substituting equations (3) into equation (1) yields:

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0$$

Expanding:

$$A\left(x'^2\cos^2\theta - 2x'y'\sin\theta\cos\theta + y'^2\sin^2\theta\right) + B\left[x'^2\sin\theta\cos\theta + x'y'\left(\cos^2\theta - \sin^2\theta\right) - y'^2\sin\theta\cos\theta\right] + C\left(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta\right) + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0$$

Now regrouping to structure as (2):

$$\left(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta\right)x'^2 + \left[-2A\sin\theta\cos\theta + B\left(\cos^2\theta - \sin^2\theta\right) + 2C\sin\theta\cos\theta\right]x'y' \left(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\right)y'^2 + (D\cos\theta + E\sin\theta)x' + (-D\sin\theta + E\cos\theta)y' + F = 0$$

So, the coefficients in (2) are related to those in (1) as:

$$A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = A\left(\frac{1+\cos 2\theta}{2}\right) + B\left(\frac{\sin 2\theta}{2}\right) + C\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= \frac{1}{2}\left\{(A+C) + \left[(A-C)\cos 2\theta + B\sin 2\theta\right]\right\}$$

$$B' = -2A\sin\theta\cos\theta + B\left(\cos^2\theta - \sin^2\theta\right) + 2C\sin\theta\cos\theta = B\cos 2\theta - (A-C)\sin 2\theta$$

$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta = A\left(\frac{1-\cos 2\theta}{2}\right) - B\left(\frac{\sin 2\theta}{2}\right) + C\left(\frac{1+\cos 2\theta}{2}\right)$$
$$= \frac{1}{2}\left\{(A+C) - \left[(A-C)\cos 2\theta + B\sin 2\theta\right]\right\}$$

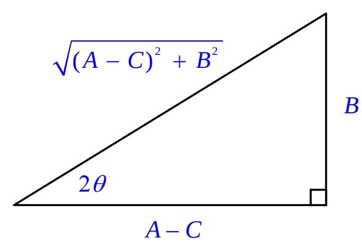
$$D' = D\cos\theta + E\sin\theta$$
$$E' = -D\sin\theta + E\cos\theta$$
$$F' = F$$

Hence, the proper choice of θ in (3) for rotation of (1) occurs when:

$$B' = B\cos 2\theta - (A-C)\sin 2\theta = 0 \quad \Rightarrow \quad \underline{\tan 2\theta = \frac{B}{A-C}} \quad \textit{or alternatively}, \quad \cot 2\theta = \frac{A-C}{B}.$$

Now, evaluate the expression $B'^2 - 4A'C'$ (notice that B' is zero after the above rotation angle):

$$B'^2 - 4A'C' = -4\cdot\frac{1}{2}\left\{(A+C) + \left[(A-C)\cos 2\theta + B\sin 2\theta\right]\right\}\cdot\frac{1}{2}\left\{(A+C) - \left[(A-C)\cos 2\theta + B\sin 2\theta\right]\right\}$$
$$= -(A+C)^2 + \left[(A-C)\cos 2\theta + B\sin 2\theta\right]^2$$
$$= -(A+C)^2 + \left[(A-C)\cdot\frac{A-C}{\sqrt{(A-C)^2+B^2}} + B\cdot\frac{B}{\sqrt{(A-C)^2+B^2}}\right]^2$$
$$= -(A+C)^2 + \left[(A-C)^2 + B^2\right]$$
$$= -A^2 - 2AC - C^2 + A^2 - 2AC + C^2 + B^2 = \boxed{B^2 - 4AC}$$



It can be concluded now that the following expressions are INVARIANT under the transformation in (3):

$$F \equiv F',$$
$$A + C \equiv A' + C', \textit{ and most significantly},$$
$$B^2 - 4AC \equiv B'^2 - 4A'C' \equiv \underline{-4A'C'} \quad (\textit{again, note } B' = 0)$$

The last expression can now be used as an indicator of the conic type (or degenerate thereof) of equation (1):

$B^2 - 4AC < 0 \;\Rightarrow\; A'C' > 0 \;\Rightarrow\;$ *A' and C' are the same sign* $\;\Rightarrow\;$ (1) *is an* <u>*ellipse*</u> ;

$B^2 - 4AC > 0 \;\Rightarrow\; A'C' < 0 \;\Rightarrow\;$ *A' and C' are of different sign* $\;\Rightarrow\;$ (1) *is a* <u>*hyperbola*</u> ;

$B^2 - 4AC = 0 \;\Rightarrow\; A'C' = 0 \;\Rightarrow\;$ *A' or C' is zero* $\;\Rightarrow\;$ (1) *is a* <u>*parabola*</u> .