Fourier Series

A set S = $\left\{ \Phi_1(x),\ \Phi_2(x),\ \cdots,\Phi_n(x) \right\}$ of n $\geq$ 2 functions is called *orthonormal* on the closed interval $[a,b]$ if $\int_a^b \Phi_i(x)\cdot\Phi_j(x)\,dx = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases} = \delta_{ij}\ (Kronecker\ delta)$. Orthonormal functions are of major importance in function estimation.

Consider the premise:

Estimate a function $f(x)$ that is piecewise continuous on the interval $[a,b]$ by another function $g(x) = \sum_{i=1}^{n} c_i\,\Phi_i(x) = c_1\Phi_1(x) + c_2\Phi_2(x) + \cdots + c_n\Phi_n(x)$, i.e., a linear combination of the functions of the given orthonormal set S.

<u>Theorem (Fourier):</u>

The error in estimation is *minimized* when $c_i = \int_a^b f(x)\,\Phi_i(x)\,dx \quad (1 \leq i \leq n)$. These coefficients c_i are called the Fourier coefficients of the function $f(x)$ for the orthonormal set S over the interval $[a, b]$, and the function $g(x)$ is therefore referred to as an n<u>th</u> order Fourier Series expansion of the function $f(x)$.

<u>Example</u>: Estimate $f(x) = \sqrt{x}$ on $[\,0,\,4\,]$ by a "best-fit" linear function $g(x) = c_1 + c_2\,x$ using the orthonormal set S = $\left\{ \Phi_1(x),\ \Phi_2(x) \right\} = \left\{ \frac{1}{2}, \frac{\sqrt{3}}{4}(x-2) \right\}$.

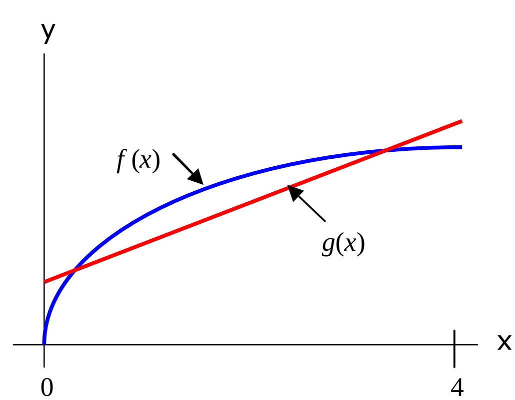
Here,

$$g(x) = \left(\int_0^4 f(x)\cdot\Phi_1(x)\,dx \right)\cdot\Phi_1(x) + \left(\int_0^4 f(x)\cdot\Phi_2(x)\,dx \right)\cdot\Phi_2(x)$$

$$= \left(\int_0^4 \sqrt{x}\cdot\frac{1}{2}\,dx \right)\cdot\frac{1}{2} + \left(\int_0^4 \sqrt{x}\cdot\frac{\sqrt{3}}{4}(x-2)\,dx \right)\cdot\frac{\sqrt{3}}{4}(x-2)$$

$$= \frac{1}{4}\left[\frac{2}{3}x^{3/2} \right]_0^4 + \frac{3}{16}\left[\frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2} \right]_0^4 (x-2)$$

$$= \boxed{\frac{8}{15} + \frac{2}{5}x}$$



Now consider the set S of $2n + 1$ functions:

$$S = \left\{ \Phi_1(x),\ \ \Phi_2(x),\ \cdots ,\Phi_{n+1}(x),\ \ \Phi_{n+2}(x),\ \cdots ,\Phi_{2n+1}(x) \right\}$$

$$= \left\{ \frac{1}{\sqrt{2L}},\ \frac{1}{\sqrt{L}}\cos\frac{\pi x}{L},\ \frac{1}{\sqrt{L}}\cos\frac{2\pi x}{L},\ \cdots ,\ \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L},\ \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L},\ \frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L},\ \cdots ,\ \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L} \right\}$$

The set S is orthonormal over the closed intervals $\left[-L, L\right]$ *or* $\left[0, 2L\right]$, again because the integrals $\int_{-L}^{L} \Phi_i(x)\cdot\Phi_j(x)\,dx = \int_{0}^{2L} \Phi_i(x)\cdot\Phi_j(x)\,dx = \delta_{ij}$. Due to the nature of these functions, this set S is of particular importance when estimating functions that are even or odd or possess a periodic property.

Notice now,

$$g(x) = c_1\Phi_1(x) + \left[c_2\Phi_2(x) + \cdots + c_{n+1}\Phi_{n+1}(x)\right] + \left[c_{n+2}\Phi_{n+2}(x) + c_{n+3}\Phi_{n+3}(x) + \cdots + c_{2n+1}\Phi_{2n+1}(x)\right]$$

$$= c_1\cdot\frac{1}{\sqrt{2L}} + \sum_{k=1}^{n} c_{k+1}\cdot\frac{1}{\sqrt{L}}\cos\left(\frac{k\pi x}{L}\right) + \sum_{k=1}^{n} c_{n+k+1}\cdot\frac{1}{\sqrt{L}}\sin\left(\frac{k\pi x}{L}\right)$$

but again,
the "best" estimate using the set S occurs if the c's are the Fourier coefficients, i.e.:

$$c_1 = \int_{-L}^{L} f(x)\cdot\frac{1}{\sqrt{2L}}\,dx\ ;$$

$$c_{k+1} = \int_{-L}^{L} f(x)\cdot\frac{1}{\sqrt{L}}\cos\left(\frac{k\pi x}{L}\right)dx\ ;$$

$$c_{n+k+1} = \int_{-L}^{L} f(x)\cdot\frac{1}{\sqrt{L}}\sin\left(\frac{k\pi x}{L}\right)dx$$

so in general,
the $n^{\underline{th}}$ *order* Fourier Series expansion of $f(x)$ *on* $[-L, L]$ can be written as:

$$g(x) = a_0 + \sum_{k=1}^{n} a_k\cos\left(\frac{k\pi x}{L}\right) + b_k\sin\left(\frac{k\pi x}{L}\right),$$

where:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\ ,$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx \quad \text{and,}$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

Example: Estimate the Heaviside function H(x) = $\begin{cases} 0 & -2 < x < 0 \\ \tfrac{1}{2} & x = 2m,\ m \in \mathbb{Z} \\ 1 & 0 < x < 2 \end{cases}$ by an n^{th} order Fourier trigonometric series using the orthonormal set:

$$S = \left\{ \frac{1}{\sqrt{2L}},\ \cdots\ \frac{1}{\sqrt{L}}\cos\frac{k\pi x}{L}\ \cdots\ ,\ \cdots\ \frac{1}{\sqrt{L}}\sin\frac{k\pi x}{L}\ \cdots \right\}_{k=1}^{n}$$ on the interval $[-2, 2]$:

Here, note that $L = 2$ and again, let:

$$g(x) = a_0 + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi x}{2}\right) + b_k \sin\left(\frac{k\pi x}{2}\right),$$

where:

$$a_0 = \frac{1}{4}\int_{-2}^{2} H(x)\,dx = \frac{1}{4}\int_{0}^{2} 1\,dx = \boxed{\frac{1}{2}}$$

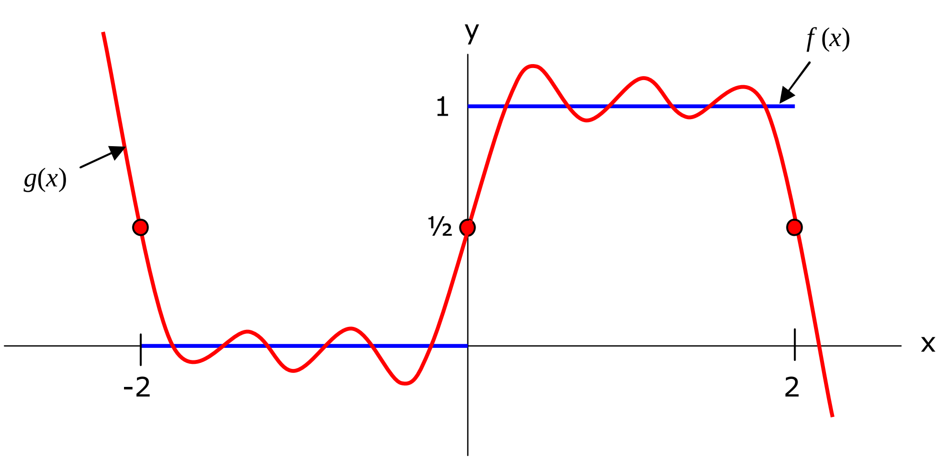
$$a_k = \frac{1}{2}\int_{-2}^{2} H(x)\cos\left(\frac{k\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} 1\cdot\cos\left(\frac{k\pi x}{2}\right)dx = \boxed{0}$$

$$b_k = \frac{1}{2}\int_{-2}^{2} H(x)\sin\left(\frac{k\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} 1\cdot\sin\left(\frac{k\pi x}{2}\right)dx = \frac{1}{k\pi}\left[1-(-1)^k\right] = \boxed{\begin{cases} \dfrac{2}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}}$$

thus,

$$g(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \cdots\right]$$

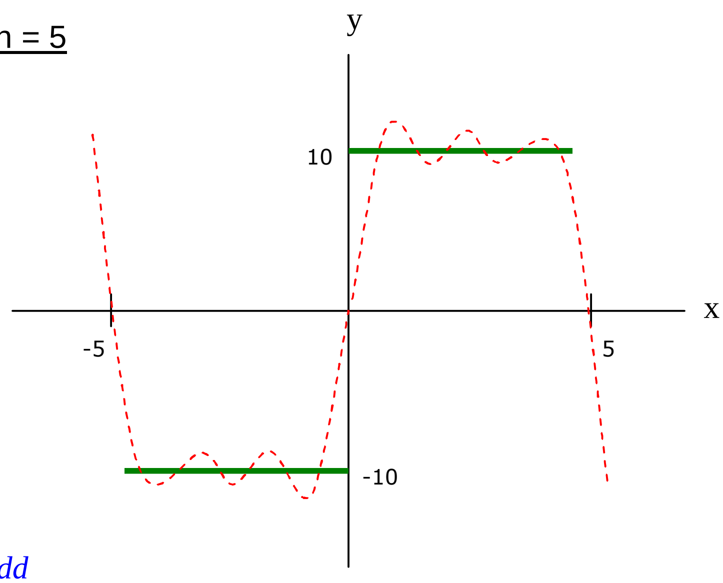
$$= \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}\cdot\sin\left[\frac{(2k-1)\pi x}{2}\right] \qquad (n = 3 \text{ shown below})$$



Examples (Odd Functions)

Calculate $g(x)$, the $\underline{n^{th}}$ order Fourier Series for $f(x)$ below and sketch the graphs of f and g:

1. $f(x) = \begin{cases} -10 & -5 < x < 0 \\ 10 & 0 < x < 5 \\ 0 & x = 0, \pm 5 \end{cases}$ $\quad \underline{n = 5}$



Here : $a_0 = a_k = 0$; $L = 5$;

$$b_k = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{k\pi x}{5}\,dx$$

$$= \frac{2}{5}\int_{0}^{5} 10\sin\frac{k\pi x}{5}\,dx$$

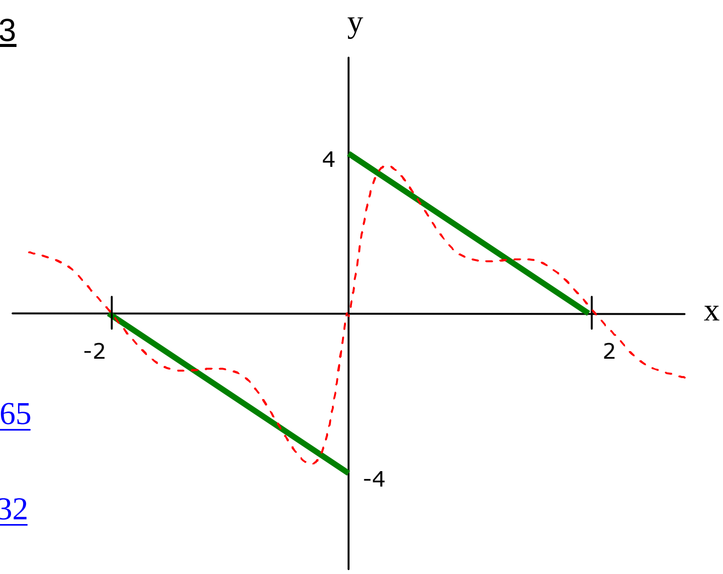
$$= \frac{2}{5}\left[-\frac{50}{k\pi}\cos\frac{k\pi x}{5}\right]_0^5$$

$$= -\frac{20}{k\pi}\left(\cos k\pi - 1\right)$$

$$= \frac{20}{k\pi}\left[1-(-1)^k\right] = \begin{cases} 40/k\pi & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$\Rightarrow g(x) = \boxed{12.7324\sin\left(\pi x/5\right) + 4.2441\sin\left(3\pi x/5\right) + 2.5465\sin\left(\pi x\right)}$$

2. $f(x) = \begin{cases} -4-2x & -2 \le x < 0 \\ 4-2x & 0 < x \le 2 \\ 0 & x = 0 \end{cases}$ $\quad \underline{n = 3}$



Here : $a_0 = a_k = 0$; $L = 2$;

$$b_k = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{k\pi x}{2}\,dx$$

$$= \frac{2}{2}\int_{0}^{2} (4-2x)\sin\frac{k\pi x}{2}\,dx$$

$$b_1 = \int_0^2 (4-2x)\sin\frac{1\pi x}{2}\,dx = \underline{2.5465}$$

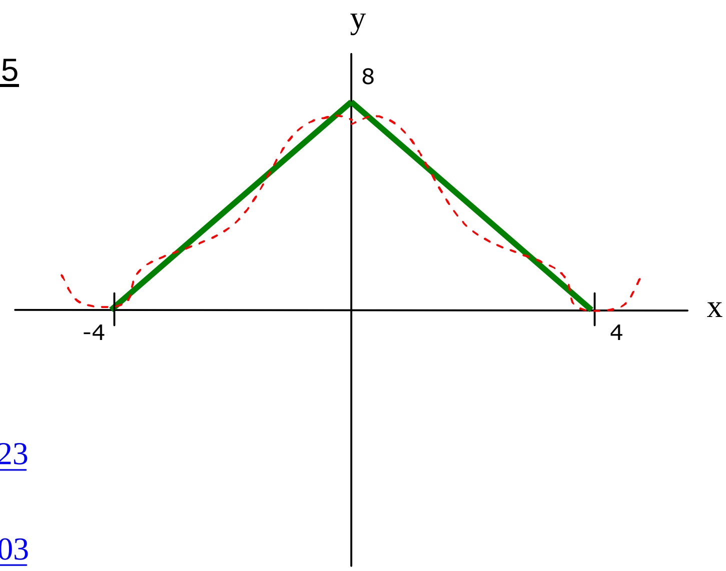
$$b_2 = \int_0^2 (4-2x)\sin\frac{2\pi x}{2}\,dx = \underline{1.2732}$$

$$b_3 = \int_0^2 (4-2x)\sin\frac{3\pi x}{2}\,dx = \underline{0.8488}$$

$$\Rightarrow g(x) = \boxed{2.5465\sin\left(\pi x/2\right) + 1.2732\sin\left(\pi x\right) + 0.8488\sin\left(3\pi x/2\right)}$$

Examples (Even Functions)

3. $f(x) = \begin{cases} 8+2x & -4 \le x < 0 \\ 8-2x & 0 < x \le 4 \\ 8 & x = 0 \end{cases}$ <u>n = 5</u>



$Here: b_k = 0 \;\; \& \;\; a_{2k} = 0 \;\; ; \;\; L = 4 \;\; ;$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{4}\int_0^4 (8-2x)\,dx = \underline{4}$$

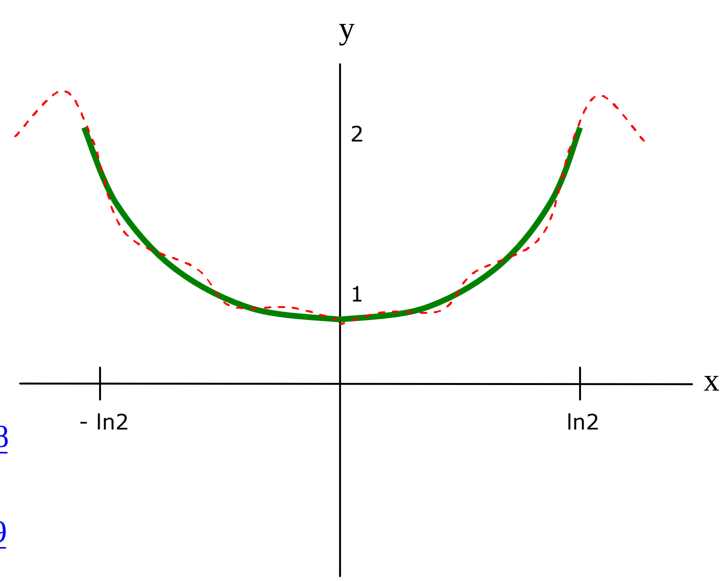
$$a_1 = \int_0^4 (4-x)\cos\frac{1\pi x}{4}\,dx = \underline{3.2423}$$

$$a_3 = \int_0^4 (4-x)\cos\frac{3\pi x}{4}\,dx = \underline{0.3603}$$

$$a_5 = \int_0^4 (4-x)\cos\frac{5\pi x}{4}\,dx = \underline{0.1297}$$

$$\Rightarrow g(x) = \boxed{4 + 3.2423\cos\left(\pi x/4\right) + 0.3603\cos\left(3\pi x/4\right) + 0.1297\cos\left(5\pi x/4\right)}$$

4. $f(x) = \begin{cases} e^{-x} & -\ln 2 \le x < 0 \\ e^{x} & 0 < x \le \ln 2 \\ 1 & x = 0 \end{cases}$ <u>n = 3</u>



$Here: b_k = 0 \;\; ; \;\; L = \ln 2 \;\; ;$

$$a_0 = \frac{1}{2\ln 2}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{\ln 2}\int_0^{\ln 2} e^x\,dx = \underline{1.4427}$$

$$a_1 = \frac{2}{\ln 2}\int_0^{\ln 2} e^x\cos\frac{1\pi x}{\ln 2}\,dx = \underline{-0.4018}$$

$$a_2 = \frac{2}{\ln 2}\int_0^{\ln 2} e^x\cos\frac{2\pi x}{\ln 2}\,dx = \underline{0.03469}$$

$$a_3 = \frac{2}{\ln 2}\int_0^{\ln 2} e^x\cos\frac{3\pi x}{\ln 2}\,dx = \underline{-0.04657}$$

$$\Rightarrow g(x) = \boxed{1.4427 - 0.4018\cos\left(\pi x/\ln 2\right) + 0.03469\cos\left(2\pi x/\ln 2\right) - 0.04657\cos\left(3\pi x/\ln 2\right)}$$