Ferdinand Georg Frobenius

Born: 26 Oct 1849 in Berlin-Charlottenburg, Prussia (now Germany)
Died: 3 Aug 1917 in Berlin, Germany

Georg Frobenius's father was Christian Ferdinand Frobenius, a Protestant parson, and his mother was Christine Elizabeth Friedrich. Georg was born in Charlottenburg which was a district of Berlin which was not incorporated into the city until 1920. He entered the Joachimsthal Gymnasium in 1860 when he was nearly eleven years old and graduated from the school in 1867. In this same year he went to the University of Göttingen where he began his university studies but he only studied there for one semester before returning to Berlin.

Back at the University of Berlin he attended lectures by Kronecker, Kummer and Weierstrass. He continued to study there for his doctorate, attending the seminars of Kummer and Weierstrass, and he received his doctorate (awarded with distinction) in 1870 supervised by Weierstrass. In 1874, after having taught at secondary school level first at the Joachimsthal Gymnasium then at the Sophienrealschule, he was appointed to the University of Berlin as an extraordinary professor of mathematics.

For the description of Frobenius's career so far, the attentive reader may have noticed that no mention has been made of him receiving his habilitation before being appointed to a teaching position. This is not an omission, rather it is surprising given the strictness of the German system that this was allowed. Details of this appointment are given in [3] but we should say that it must ultimately have been made possible due to strong support from Weierstrass who was extremely influential and considered Frobenius one of his most gifted students.

Frobenius was only in Berlin for a year before he went to Zürich to take up an appointment as an ordinary professor at the Eidgenössische Polytechnikum. For seventeen years, between 1875 and 1892, Frobenius worked in Zürich. He married there and brought up a family and did much important work in widely differing areas of mathematics. We shall discuss some of the topics which he worked on below, but for the moment we shall continue to describe how Frobenius's career developed.

In the last days of December 1891 Kronecker died and, therefore, his chair in Berlin became vacant. Weierstrass, strongly believing that Frobenius was the right person to keep Berlin in the forefront of mathematics, used his considerable influence to have Frobenius appointed. However, for reasons which we shall discuss in a moment, Frobenius turned out to be something of a mixed blessing for mathematics at the University of Berlin.

The positive side of his appointment was undoubtedly his remarkable contributions to the representation theory of groups, in particular his development of character theory, and his position as one of the leading mathematicians of his day. The negative side came about largely through his personality which is described in [5] as:-

... occasionally choleric, quarrelsome, and given to invectives.

Biermann, in [3], looks more closely at his character (no pun intended!), and how it affected the success of mathematical education at the university. He describes the strained relationships which developed between Frobenius and his colleagues at Berlin. He had such high standards that in the end these did not serve Berlin well. He [3]:-

... suspected at every opportunity a tendency of the Ministry to lower the standards at the University of Berlin, in the words of Frobenius, to the rank of a technical school ... Even so, Fuchs and Schwarz yielded to him, and later Schottky, who was indebted to him alone for his call to Berlin. Frobenius was the leading figure, on whom the fortunes of mathematics at Berlin university rested for 25 *years. Of course, it did not escape him, that the number of doctorates, habilitations, and docents slowly but surely fell off, although the number of students increased considerably. That he could not prevent this, that he could not reach his goal of maintaining unchanged the times of Weierstrass, Kummer and Kronecker also in their external appearances, but to witness helplessly these developments, was doubly intolerable for him, with his choleric disposition.*

We should not be too hard on Frobenius for, as Haubrich explains in [5], Frobenius's attitude was one which was typical of all professors of mathematics at Berlin at this time:-

They all felt deeply obliged to carry on the Prussian neo-humanistic tradition of university research and teaching as they themselves had experienced it as students. This is especially true of Frobenius. He considered himself to be a scholar whose duty it was to contribute to the knowledge of pure mathematics. Applied mathematics, in his opinion, belonged to the technical colleges.

The view of mathematics at the University of Göttingen was, however, very different. This was a time when there was competition between mathematians in the University of Berlin and in the University of Göttingen, but it was a competition that Göttingen won, for there mathematics flourished under Klein, much to Frobenius's annoyance. In [3] Biermann writes that:-

The aversion of Frobenius to Klein and S Lie knew no limits ...

Frobenius hated the style of mathematics which Göttingen represented. It was a new approach which represented a marked change from the traditional style of German universities. Frobenius, as we said above, had extremely traditional views. In a letter to Hurwitz, who was a product of the Göttingen system, he wrote on 3 February 1896 (see [4]):-

If you were emerging from a school, in which one amuses oneself more with rosy images than hard ideas, and if, to my joy, you are also gradually becoming emancipated from that, then old loves don't rust. Please take this joke facetiously.

One should put the other side of the picture, however, for in [9] Siegel, who knew Frobenius for two years from 1915 when he became a student until Frobenius's death, relates his impression of Frobenius as having a warm personality and expresses his appreciation of his fast-paced varied and deep lectures. Others would describe his lectures as solid but not stimulating.

To gain an impression of the quality of Frobenius's work before the time of his appointment to Berlin in 1892 we can do no better than to examine the recommendations of Weierstrass and Fuchs when Frobenius was elected to the Prussian Academy of Sciences in 1892. Fairly extensive quotes from this document, and another similar document from Fuchs and Helmholtz, are given in [4] but we quote a short extract to show the power, variety and high quality of Frobenius's work in his Zürich years. Weierstrass and Fuchs list 15 topics on which Frobenius had made major contributions:-

1. *On the development of analytic functions in series.*
2. *On the algebraic solution of equations, whose coefficients are rational functions of one variable.*
3. *The theory of linear differential equations.*
4. *On Pfaff's problem.*
5. *Linear forms with integer coefficients.*
6. *On linear substitutions and bilinear forms...*
7. *On adjoint linear differential operators...*
8. *The theory of elliptic and Jacobi functions...*

9. *On the relations among the* 28 *double tangents to a plane of degree 4.*
10. *On Sylow's theorem.*
11. *On double cosets arising from two finite groups.*
12. *On Jacobi's covariants...*
13. *On Jacobi functions in three variables.*
14. *The theory of biquadratic forms.*
15. *On the theory of surfaces with a differential parameter.*

In his work in group theory, Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups. He published *Über Gruppen von vertauschbaren Elementen* in 1879 (jointly with Stickelberger, a colleague at Zürich) which looks at permutable elements in groups. This paper also gives a proof of the structure theorem for finitely generated abelian groups. In 1884 he published his next paper on finite groups in which he proved Sylow's theorems for abstract groups (Sylow had proved his theorem as a result about permutation groups in his original paper). The proof which Frobenius gives is the one, based on conjugacy classes, still used today in most undergraduate courses.

In his next paper in 1887 Frobenius continued his investigation of conjugacy classes in groups which would prove important in his later work on characters. In the introduction to this paper he explains how he became interested in abstract groups, and this was through a study of one of Kronecker's papers. It was in the year 1896, however, when Frobenius was professor at Berlin that his really important work on groups began to appear. In that year he published five papers on group theory and one of them *Über die Gruppencharactere* on group characters is of fundamental importance. He wrote in this paper:-

I shall develop the concept [*of character for arbitrary finite groups*] *here in the belief that through its introduction, group theory will be substantially enriched.*

This paper on group characters was presented to the Berlin Academy on July 16 1896 and it contains work which Frobenius had undertaken in the preceding few months. In a series of letters to Dedekind, the first on 12 April 1896, his ideas on group characters quickly developed. Ideas from a paper by Dedekind in 1885 made an important contribution and Frobenius was able to construct a complete set of representations by complex numbers. It is worth noting, however, that although we think today of Frobenius's paper on group characters as a fundamental work on representations of groups, Frobenius in fact introduced group characters in this work without any reference to representations. In was not until the following year that representations of groups began to enter the picture, and again it was a concept due to Frobenius. Hence 1897 is the year in which the representation theory of groups was born.

Over the years 1897-1899 Frobenius published two papers on group representations, one on induced characters, and one on tensor product of characters. In 1898 he introduced the notion of induced representations and the Frobenius Reciprocity Theorem. It was a burst of activity which set up the foundations of the whole of the machinery of representation theory.

In a letter to Dedekind on 26 April 1896 Frobenius gave the irreducible characters for the alternating groups A_4, A_5, the symmetric groups S_4, S_5 and the group $PSL(2,7)$ of order 168. He completely determined the characters of symmetric groups in 1900 and of characters of alternating groups in 1901, publishing definitive papers on each. He continued his applications of character theory in papers of 1900 and 1901 which studied the structure of Frobenius groups.

Only in 1897 did Frobenius learn of Molien's work which he described in a letter to Dedekind as "very beautiful but difficult". He reformulated Molien's work in terms of matrices and then showed that his characters are the traces of the irreducible representations. This work was published in 1897. Frobenius's character theory was used with great effect by Burnside and was beautifully written up in Burnside's 1911 edition of his *Theory of Groups of Finite Order.*

Frobenius had a number of doctoral students who made important contributions to mathematics. These included Edmund Landau who was awarded his doctorate in 1899, Issai Schur who was awarded his doctorate in 1901, and Robert Remak who was awarded his doctorate in 1910. Frobenius collaborated with Schur in representation theory of groups and character theory of groups. It is certainly to Frobenius's credit that he so quickly spotted the genius of his student Schur. Frobenius's representation theory for finite groups was later to find important applications in quantum mechanics and theoretical physics which may not have entirely pleased the man who had such "pure" views about mathematics.

Among the topics which Frobenius studied towards the end of his career were positive and non-negative matrices. He introduced the concept of irreducibility for matrices and the papers which he wrote containing this theory around 1910 remain today the fundamental results in the discipline. The fact so many of Frobenius's papers read like present day text-books on the topics which he studied is a clear indication of the importance that his work, in many different areas, has had in shaping the mathematics which is studied today. Having said that, it is also true that he made fundamental contributions to fields which had already come into existence and he did not introduce any totally new mathematical areas as some of the greatest mathematicians have done.

In [5] Haubrich gives the following overview of Frobenius's work:-

The most striking aspect of his mathematical practice is his extraordinary skill at calculations. In fact, Frobenius tried to solve mathematical problems to a large extent by means of a calculative, algebraic approach. Even his analytical work was guided by algebraic and linear algebraic methods. For Frobenius, conceptual argumentation played a somewhat secondary role. Although he argued in a comparatively abstract setting, abstraction was not an end in itself. Its advantages to him seemed to lie primarily in the fact that it can lead to much greater clearness and precision.

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