TI-84/83 graphing calculator

Let us use the table feature to approximate limits and then solve the problems the old fashion way.

Find the following limits:

Example 1.1 $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ $f(x) = \dfrac{x^2 - 4}{x - 2}$

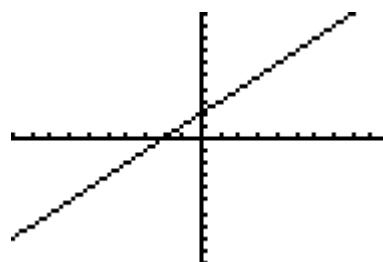Turn on your TI-84 graphing calculator.

Press the $\boxed{\text{Y=}}$ tab.
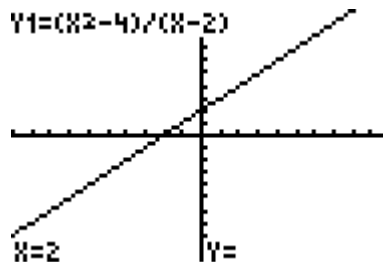
Type $(x^2 - 4)/(x - 2)$ on the first line after
\Y1=.

```
Plot1 Plot2 Plot3
\Y1目(X²-4)/(X-2)
■
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Press $\boxed{\text{Zoom}}$ followed by number $\boxed{6}$ (to make sure you are in a standard window).

Press $\boxed{\text{TRACE}}$ followed by number $\boxed{2}$ and then $\boxed{\text{ENTER}}$.

Note that, $Y = f(2)$ does not exist. But, what about $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$?

1

Let us use the calculator. Press **2ND** key followed by **WINDOW**.

Let us use the right and left approach. Using the up and down arrows:

type after TblStart $=1.997$, and
type after ΔTbl $=0.001$.

```
TABLE SETUP
 TblStart=1.997
 ⌐Tbl=0.001■
Indent: ████ Ask
Depend: ████ Ask
```

Make sure both Indepnt: and Depend: **Auto** is highlighted by using the left and right arrows and then pressing **ENTER**.

Now press Press **2ND** key followed by **GRAPH** to see the table.

Note that the table agrees with our conclusions that the $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}=4$,

and that $f(2)=DNE$.

```
  X    │ Y1  │
██████ │3.997│
 1.998 │3.998│
 1.999 │3.999│
 2     │ERR: │
 2.001 │4.001│
 2.002 │4.002│
 2.003 │4.003│
X=1.997
```

Example 1.2 $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 7 & \text{if } x = 2 \end{cases}$

Find all values of x where f is continuous.

Press the **Y=** tab.

Type $(x^2-4)/(x-2)/(x \neq 2)$ on the first line after \Y1=.

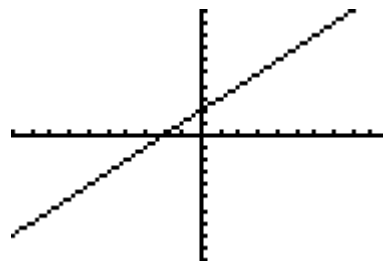(Note: Press **2nd** followed by **MATH** key and then **2** key to see the $\neq$ sign).

Type $7/(x=2)$ on the second line after \Y2=.

(Note: Press **2nd** followed by **MATH** key and

```
Plot1 Plot2 Plot3
\Y1⌐(X²-4)/(X-2)
/(X≠2)
\Y2⌐7/(X=2■
\Y3=
\Y4=
\Y5=
\Y6=
```

then $\boxed{1}$ key to see the $=$ sign).

Press $\boxed{\textbf{Zoom}}$ followed by number $\boxed{6}$ (to make sure you are in a standard window).

Note that $f(2)$ exists and is equal to 7. To see this, press $\boxed{\textbf{TRACE}}$ followed by the upward arrow to change the graph from Y1 to Y2. (Note that the left corner number changes from Y1 to Y2). Once you see the equation Y2 in the upper left corner, press the number $\boxed{2}$ followed by $\boxed{\textbf{ENTER}}$ to see that $f(2) = 7$.
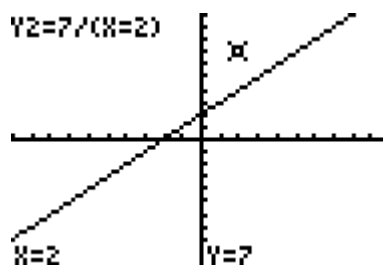
To show that the $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$,

type after TblStart $= 1.997$, and
type after ΔTbl $= 0.001$.

Make sure both Indepnt: and Depend: $\boxed{\textbf{Auto}}$ is highlighted by using the left and right arrows and then pressing $\boxed{\textbf{ENTER}}$.

Now press $\boxed{\textbf{2ND}}$ followed by $\boxed{\textbf{GRAPH}}$ to see the table.

Note that the table agrees with our conclusions that the $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$ and that $f(2) = 7$.

So this function is continuous for all real numbers x except when $x = 2$.

3

However, this function $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$ is continuous on the entire real line.

Example 1.3 $f(x) = \dfrac{4}{x - 2}$

Find all vertical and horizontal asymptotes to the graph of f.

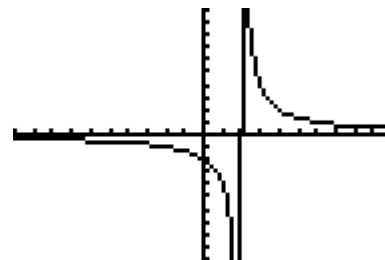Press the $\boxed{\text{Y=}}$ tab.

Type $4/(x - 2)$ on the first line after \Y1=.

Press the $\boxed{\textbf{GRAPH}}$ tab. (Make sure you are in a standard window).

Note that $f(2)$ does not exist. To see this, press $\boxed{\text{TRACE}}$ followed by the $\boxed{2}$ key, and then $\boxed{\text{ENTER}}$.
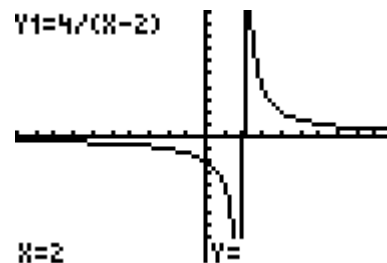
Let us use the left and right approach to find the limit. Using the up and down arrows: Press $\boxed{\textbf{2ND}}$ followed by the $\boxed{\textbf{WINDOW}}$ tab

type after TblStart $= 1.9997$, and type after ΔTbl $= 0.0001$.

4

Make sure both Indepnt: and Depend: **Auto** is highlighted by using the left and right arrows and then pressing **ENTER**.

To see the table, Press **2ND** followed by **GRAPH** tab.

X	Y₁
1.997	-1333
1.998	-2000
1.999	-4000
2	ERR:
2.001	4000
2.002	2000
2.003	1333.3

X=1.997

Therefore, $\lim\limits_{x \to 2^-} \dfrac{4}{x-2} = -\infty$ and $\lim\limits_{x \to 2^+} \dfrac{4}{x-2} = +\infty$. That is, $x = 2$ is a vertical asymptote.

Next let us find the Horizontal asymptote. Press the **2ND** key followed by **WINDOW**. Using the up and down arrows:

type after $\text{TblStart} = -600$, and type after $\Delta\text{Tbl} = -600$.

TABLE SETUP
 TblStart=-600
 △Tbl=-600█
Indent: **Auto** Ask
Depend: **Auto** Ask

Make sure both Indepnt: and Depend: **Auto** is highlighted by using the left and right arrows and then pressing **ENTER**.

To see the table, Press **2ND** followed by **GRAPH** tab.

X	Y₁
-600	-.0066
-1200	-.0033
-1800	-.0022
-2400	-.0017
-3000	-.0013
-3600	-.0011
-4200	-1E-3

X= -600

This means $\lim\limits_{x \to -\infty} \dfrac{4}{x-2} = 0$.

Similarly,

Press the **2ND** key followed by **WINDOW**. Using the up and down arrows:

type after $TblStart = 600$, and

type after $\Delta Tbl = 600$.

Make sure both Indepnt: and Depend: **Auto** is highlighted by using the left and right arrows and then pressing **ENTER**.

```
TABLE SETUP
 TblStart=600
 △Tbl=600■
Indpnt: Auto Ask
Depend: Auto Ask
```

To see the table, Press **2ND** followed by **GRAPH** tab.

```
  X   | Y₁  |
 600  |.00669|
 1200 |.00334|
 1800 |.00222|
 2400 |.00167|
 3000 |.00133|
 3600 |.00111|
 4200 |9.5E⁻⁴|
X=600
```

This means $\lim\limits_{x \to +\infty} \dfrac{4}{x-2} = 0$.

Therefore, $\lim\limits_{x \to \pm\infty} \dfrac{4}{x-2} = 0$, and $y = 0$ is a horizontal asymptote.

Example 1.4 Given $f(x) = 0.2x^3 - 3x^2 + 4x + 7.4$

Find $\delta > 0$ such that $\lim\limits_{x \to 2}\left(0.2x^3 - 3x^2 + 4x + 7.4\right) = 5$ within 2 decimal places.

In this case, we need to find a $\delta > 0$ such that $|x - 2| < \delta$ implies $\left|\left(0.2x^3 - 3x^2 + 4x + 7.4\right) - 5\right| < 0.01$. We will use the calculator.
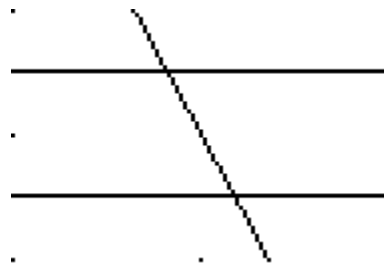
Press the **Y=** tab.

Type 4.99 on the first line after \Y1=, $0.2x^3 - 3x^2 + 4x + 7.4$ on the second line after \Y2=, and 5.01 on the third line after \Y3=.

```
Plot1 Plot2 Plot3
\Y1■4.99
\Y2■0.2X^3-3X²+4
X+7.4
\Y3■5.01
\Y4=■
\Y5=
\Y6=
```

6

```
WINDOW
 Xmin=1.99
 Xmax=2.01
 Xscl=.01
 Ymin=4.98
 Ymax=5.02
 Yscl=.01
 Xres=1█
```

Use the following window, then press **GRAPH**.

In order to find the left and right x-coordinates on the points of intersection, we use the following.

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

Press the **2ND** key followed by **TRACE**. Using the up and down arrows to select 5: intersect or basically press 5.

The up and down arrow will make the cursor jump between graphs, while the right and left arrow make the cursor move along a graph. Take the cursor to the first point of intersection, and then press **ENTER**.

```
Y3=5.01
First curve?
X=1.9982979  .Y=5.01
```

```
Y2=0.2X^3-3X²+4X+7.4
Second curve?
X=1.9980851  .Y=5.0107168 .
```

The up and down arrow will take the cursor back to the same point. Once the cursor is over the same points, you need to press **ENTER**. If the cursor is away from the point, use the right and left arrow to move as close as possible to the point before pressing **ENTER**.

```
Y2=0.2X^3-3X²+4X+7.4
Guess?
X=1.9980851  .Y=5.0107168 .
```
Then press **ENTER** to find the point of intersection.

```
Intersection
X=1.9982133  .Y=5.01         .
```
Thus $x_{left} = 1.9982133$

In order to find the value of x_{right}, we repeat the same steps to the other point of intersection.

Press the **2ND** key followed by **TRACE**. Using the up and down arrows to select 5: intersect or basically press 5.
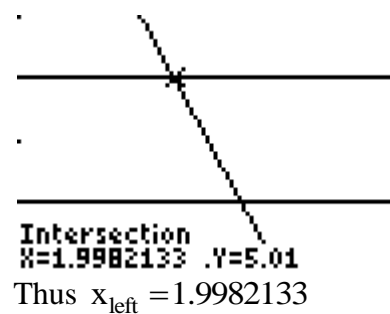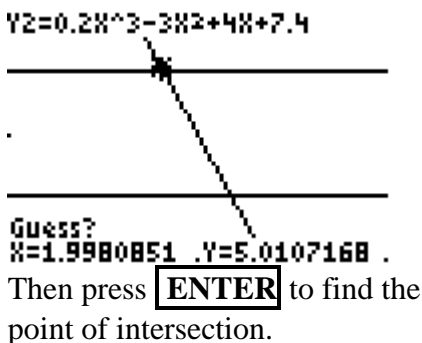
```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

The up and down arrow will make the cursor jump between graphs, while the right and left arrow make the cursor move along a graph. Take the cursor to the first point of intersection, and then press **ENTER**.

Y1=4.99

First curve?
X=1.9982979   Y=4.99

Y2=0.2X^3-3X2+4X+7.4

Second curve?
X=2.0017021   Y=4.9904629

The up and down arrow will take the cursor back to the same point. Once the cursor is over the same points, you need to press **ENTER**. If the cursor is away from the point, use the right and left arrow to move as close as possible to the point before pressing **ENTER**.
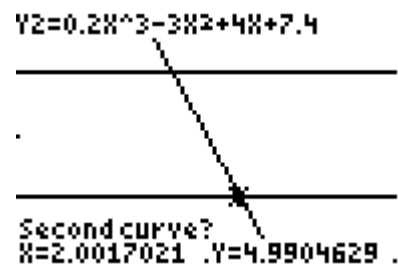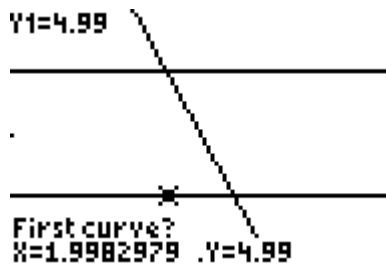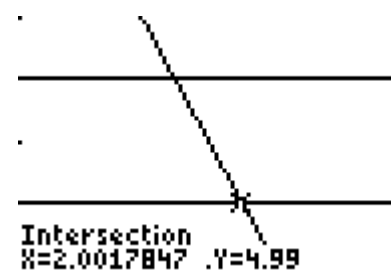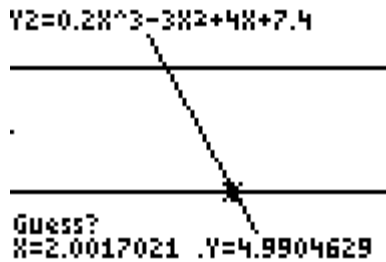
Y2=0.2X^3-3X2+4X+7.4

Guess?
X=2.0017021   Y=4.9904629

Then press **ENTER** to find the point of intersection.

Intersection
X=2.0017847   Y=4.99

Thus $x_{right} = 2.0017847$

The value of δ is the minimum of $\{2-1.9982133 ,\ 2.0017847-2\}$

$$= \text{minimum of } \{0.0017867 ,\ 0.0017847\}$$
$$= 0.0017847$$

1. **Find the derivative of a differentiable function.**

   Example 2.1   Find the slope of $f(x) = x^2$ at the point $(2,4)$.

   We know that $f'(2) = \lim_{x \to 2} \dfrac{f(x)-f(2)}{x-2}$ provided the limit exists.

   Here $f'(2) = \lim_{x \to 2} \dfrac{x^2-4}{x-2}$.

8

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Press the **Y=** tab.

Type $(x^2 - 4)/(x - 2)$ on the first line after \Y1=.

```
Plot1 Plot2 Plot3
\Y1◼(X²-4)/(X-2)
■
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Press the **2ND** key followed by **WINDOW**. Using the up and down arrows:

type after $\text{TblStart} = 1.997$, and
type after $\Delta\text{Tbl} = 0.001$.

Make sure both Indepnt: and Depend: **Auto** is highlighted by using the left and right arrows and then pressing **ENTER**.

```
TABLE SETUP
 TblStart=1.997
 ∆Tbl=0.001■
Indpnt: Auto Ask
Depend: Auto Ask
```

To see the table, Press **2ND** followed by **GRAPH** tab.

Note that the table agrees with our conclusions that the $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$.

```
  X     Y1
1.997  3.997
1.998  3.998
1.999  3.999
2      ERR:
2.001  4.001
2.002  4.002
2.003  4.003
X=1.997
```

This means that the slope $f'(2) = 4$.

Another method consists of using the following:

To go back to the main screen, press **2ⁿᵈ** followed by **MODE**, and then **ENTER**. Type **MATH** followed by **8** to see the nDeriv function, then type **nDeriv**$(x^2, x, 2)$, followed by **ENTER**.

```
nDeriv(X²,X,2)
              4
■
```

This also shows that the slope $f'(2) = 4$.

9

Example 2.2 Find the slope of $f(x) = |x - 2|$ at the point $(2, 0)$.

We know that $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ provided the limit exists.

Here $f'(2) = \lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$.

Press the **Y=** tab.

Type $\dfrac{|x - 2|}{x - 2}$ on the first line after
\Y1=.

```
Plot1 Plot2 Plot3
\Y1◘abs(X-2)/(X-
2)■
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Type **MATH** followed by the
right arrow to choose **NUM**
followed by **1** to see the abs
function, then type
$abs(x - 2)/(x - 2)$.

Press the **2ND** key followed by **WINDOW**.
Using the up and down arrows:

type after TblStart $= 1.997$, and
type after ΔTbl $= 0.001$.

```
TABLE SETUP
 TblStart=1.997
 ΔTbl=0.001■
Indpnt: Auto Ask
Depend: Auto Ask
```

Make sure both Indepnt: and Depend: **Auto**
is highlighted by using the left and right
arrows and then pressing **ENTER**.

To see the table, Press **2ND** followed by
GRAPH tab.

Note that the table agrees with our
conclusions that the $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$ does not
exist.

X	Y1	
1.997	-1	
1.998	-1	
1.999	-1	
2	ERR:	
2.001	1	
2.002	1	
2.003	1	

X=1.997

This means that the slope $f'(2)$ is undefined and that the function $f(x) = |x-2|$ is not differentiable at the point $(2,0)$.

Another **misleading** method consists of using the following:

To go back to the main screen, press **2ⁿᵈ** followed by **MODE**, and then **ENTER**. Type **MATH** followed by **8** to see the nDeriv function, then type **nDeriv**$(\text{abs}(x-2), x, 2)$, followed by **ENTER**.

```
nDeriv(abs(X-2),
X,2)
                    0
█
```

Well, the slope $f'(2)$ is still undefined, and the function $f(x) = |x-2|$ is still not differentiable at the point $(2,0)$; however, the calculator's answer is 0, which is wrong.

Example 2.3 Find the slope of $f(x) = \sin(x-2)$ at the point $(2,0)$

We know that $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x)-f(2)}{x-2}$ provided the limit exists.

Here $f'(2) = \lim\limits_{x \to 2} \dfrac{\sin(x-2)}{x-2}$.

To start, make sure your calculator's mode is in radians.

Press **MORE**, then use the up and down key to highlight Radian followed by **ENTER**.

```
NORMAL  SCI  ███
FLOAT  0 1 2 3 4 5 6 7 8 9
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
SET CLOCK 01/13/07 7:43PM
```

To set up the calculator clock, highlight the SET CLOCK selection while the calculator is in **MODE** Window, then press **ENTER**. Then change the next window to the desired time. At the end, highlight SAVE, then press **ENTER** to save the time.

```
FORMAT: M/D/Y  D/M/Y  Y/M/D
YEAR:   2007
MONTH:  1
DAY:    13
TIME:   12HOUR  24HOUR
HOUR:   7
MINUTE: 44
AM/PM:  AM  PM
███
```

Press the **Y=** tab.

```
Plot1 Plot2 Plot3
\Y1■sin(X-2)/(X-
2)■
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Type $\dfrac{\sin(x-2)}{x-2}$ on the first line
after \Y1=.

Press the **2ND** key followed by **WINDOW**.
Using the up and down arrows:

type after TblStart $=1.997$, and
type after ΔTbl $=0.001$.

```
TABLE SETUP
 TblStart=1.997
 ΔTbl=0.001■
Indpnt: Auto Ask
Depend: Auto Ask
```

Make sure both Indepnt: and Depend: **Auto**
is highlighted by using the left and right
arrows and then pressing **ENTER**.

To see the table, Press **2ND** followed by
GRAPH tab.

Note that the table agrees with our
conclusions that the $\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x-2}=1$.

```
   X    │ Y1  │
━━━━━━━━┿━━━━━┿━━━
1.997   │ 1   │
1.998   │ 1   │
1.999   │ 1   │
2       │ ERR:│
2.001   │ 1   │
2.002   │ 1   │
2.003   │ 1   │
━━━━━━━━┷━━━━━┷━━━
X=1.997
```

This also shows that the slope $f'(2)=1$, and the function value $f(2)=1$ does is undefined.

Example 2.4 Not all functions are differentiable. Consider $f(x)=x^2+1$ and
$g(x)=|x|+1$.

We know f is differentiable on the entire line, while g in not differentiable at the point
(0,1). Let us show the essence of differentiability using the graphing calculator.

Press the **Y=** tab.
Type x^2+1 on the first line
after \Y1= and $abs(x)+1$ on the
second line after \Y2= . To see
the **abs** key, Press **MATH**
followed by **NUM** using the
right and left arrows. Use the
window

```
Plot1 Plot2 Plot3
\Y1■X²+1
\Y2■abs(X)+1
\Y3=■
\Y4=
\Y5=
\Y6=
\Y7=
```

Now press **ZOOM** followed
by **ZOOM In** and then the up
or down arrows till the cursor
is over the point (0,1) and then
press **ENTER**.

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```
WINDOW
 Xmin=-1■
 Xmax=1
 Xscl=1
 Ymin=-1
 Ymax=4
 Yscl=1
 Xres=1
```

X=0   Y=1.0967742

If we repeat the last steps few
times, we find out that the graph
of f approaches the graph of a
horizontal line, while the graph
of g keeps the same shape. Look
at your window at this point and
see the xmin and the xmax.

X=0   Y=1.016129   X=0   Y=1.0060484

2. **Use limits, derivatives, critical numbers, extrema, and points of inflection to graph a function**

Example 3.1   Graph $f(x) = \dfrac{x-2}{x^2}$.

Students can easily graph this function, but most of them will not graph it as shown below and especially if a standard window is used.
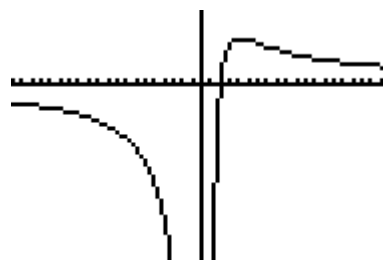
Set up the Window as follows:

```
WINDOW
 Xmin=-20
 Xmax=20
 Xscl=1
 Ymin=-.5
 Ymax=.2
 Yscl=1
 Xres=1■
```

Then we ha the following graph
for \Y1=

So let us go through the important steps:

Here, the domain of the function is $(-\infty, 0) \cup (0, \infty)$ and the x-intercept is $(2, 0)$.

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Since the $\lim_{x\to 0^-}\dfrac{x-2}{x^2}=-\infty$ and $\lim_{x\to 0^+}\dfrac{x-2}{x^2}=+\infty$, $x=0$ is a vertical asymptote.

Also since $\lim_{x\to\pm\infty}\dfrac{x-2}{x^2}=0$, $y=0$ is a horizontal asymptote.

Next, let us find the critical numbers and the possible points of inflections.

Here, $\qquad f'(x)=\dfrac{-x+4}{x^3}\qquad$ and $\qquad f''(x)=\dfrac{2(x-6)}{x^4}$

So $x=4$ is a critical number and $(6,\frac{1}{9})$ is a point of inflection.

Now let us enter $f(x)$, $f'(x)$, $f''(x)$ in the calculator.

Press the **Y=** tab.
After $\backslash$Y1=, type $(x-2)/x^2$.
After $\backslash$Y2=, type **MATH**, followed by **8**, then **VARS**, followed **Y-VARS** then **1**, and **1** again. Type in the comma **,** followed by **X,T,θ,n**, followed by comma **,** and **X,T,θ,n** one more time.
After $\backslash$Y3=, type **MATH**, followed by **8**, then **VARS**, followed **Y-VARS** then **1**, and then **2** this time. Type in the comma **,** followed by **X,T,θ,n**, followed by comma **,** and **X,T,θ,n** one more time.

```
VARS Y-VARS
1:Window…
2:Zoom…
3:GDB…
4:Picture…
5:Statistics…
6:Table…
7:String…
```

```
VARS Y-VARS
1:Function…
2:Parametric…
3:Polar…
4:On/Off…
```

```
FUNCTION
1:Y1
2:Y2
3:Y3
4:Y4
5:Y5
6:Y6
7↓Y7
```

```
Plot1  Plot2  Plot3
\Y1◨(X-2)/X²
\Y2◨nDeriv(Y1,X,
X)
\Y3◨nDeriv(Y2,X,
X)■
\Y4=
\Y5=
```

Since $x=4$ is a critical number and $(6,\frac{1}{9})$ is a point of inflection, we need to pick four test numbers different from 0, 4 and 6 (Note that 0 is not in the domain of f). How about $x=-2$, $x=2$, $x=5$, and $x=8$?

Let us set up the table.

Press the **2ND** key followed by **WINDOW**. Using the up and down arrows:

type after $\text{TblStart} = 0$, and type after $\Delta \text{Tbl} = 1$.

Make sure the Indepnt: **Ask** is highlighted   and Depend: is on **Auto** by using the left and right arrows   and   then   pressing **ENTER**.

To see the table, Press **2ND** followed by **GRAPH** tab. If there are entries in table, use the **DEL** key to delete the numbers in the x column. By deleting these numbers you clear the table.

Use the right arrow to see the Y3 column.

```
TABLE SETUP
 TblStart=0
 ⌂Tbl=1█
Indpnt: Auto ASK
Depend: AUTO Ask
```

X	Y₁	Y₂	X	Y₂	Y₃
██					██
X=			Y₃=		

Type in the following numbers in the X column. To move to the next number, you can use the downward arrow key, or simply press **ENTER**. Use the right arrow to see the Y3 column.

X	Y₁	Y₂	X	Y₂	Y₃
-2	-1	-.75	-2	-.75	-1
2	0	.25	2	.25	-.5
5	.12	-.008	5	-.008	-.0032
8	.09375	-.0078	8	-.0078	█████
X=			Y₃=9.765625ᴇ-4		

The signs under Y2 show that $f'(x) < 0$ on $(-\infty, 0) \cup (4, \infty)$ and $f'(x) > 0$ on $(0, 4)$. So $f(x)$ is decreasing $(-\infty, 0) \cup (4, \infty)$ and increasing on $(0, 4)$.
Since the first derivative changes signs from positive to negative at $(4, 0.125)$, this point is a local Maximum.

The signs under Y3 show that $f''(x) < 0$ on $(-\infty, 0) \cup (0, 6)$ and $f''(x) > 0$ on $(6, \infty)$. So $f(x)$ Concaves downwards $(-\infty, 0) \cup (0, 6)$ and upwards on $(6, \infty)$.
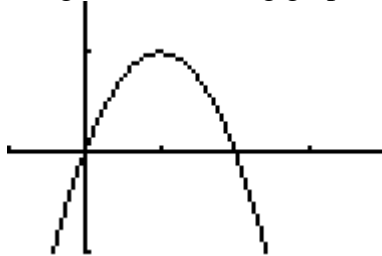Since the second derivative changes signs at $(6, 0.111)$, this point is a point of inflection.

Example 3.2 Graph $f(x) = 1 - (x-1)^2 - \dfrac{1}{1000(x-1)^{1/7} + 1}$, then find the absolute maximum on
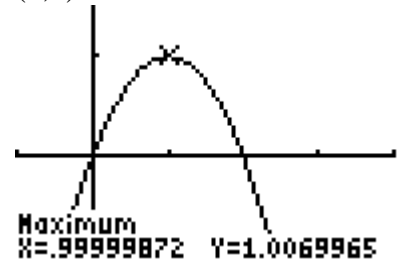
$[-1, 4]$.

If we use the Window	We get the following graph	And hence the absolute max is at (1,1).

```
WINDOW
 Xmin=-1■
 Xmax=4
 Xscl=1
 Ymin=-1
 Ymax=1.5
 Yscl=1
 Xres=1
```

```
Maximum
X=.99999872  Y=1.0069965
```

But, if we use the Window	We get the following graph	And hence the function value is 0 when $x = 1$.

```
WINDOW
 Xmin=-1
 Xmax=3■
 Xscl=1
 Ymin=-1
 Ymax=1.5
 Yscl=1
 Xres=1
```

```
Y1=1-(X-1)2-1/(1000(X-1)_
```

```
X=1          Y=0
```
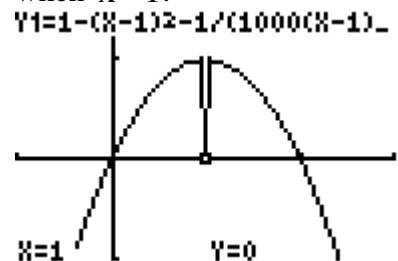
Hence, the calculator produced the wrong absolute maximum.

3. **Finding the area under the graph of a continuous function and the fundamental theorem of calculus.**

Suppose we are given a continuous function $f(x) \geq 0$ on a closed interval $[a,b]$. One way to approximate the area under $f(x)$ is by using left L_n and right R_n sums. Partition $[a,b]$ into n equal subintervals $a \leq x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n \leq b$ and let $x_i = a + i\Delta x$ where $\Delta x = \dfrac{b-a}{n}$. The left sum formula is $L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$, the right sum formula is
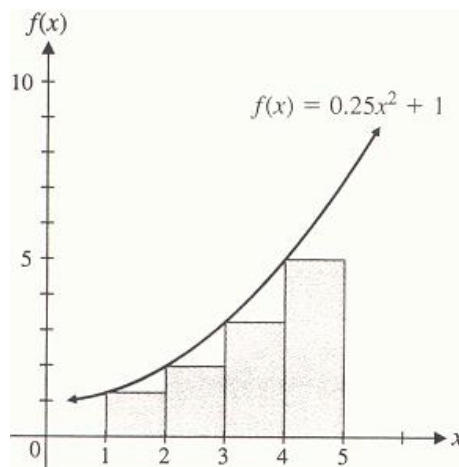
$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$, and the midpoint sum formula is $M_n = \sum_{i=1}^{n} f\left(\dfrac{x_{i-1} + x_i}{2}\right) \cdot \Delta x$.

Example 4.1 Find the sum of the areas of the left rectangles as shown in the figure.

Here $\Delta x = \dfrac{5-1}{4} = 1$ and $x_i = 1 + i(1) = 1 + i$.

So,

$$L_4 = \sum_{i=1}^{4} f(x_{i-1}) \cdot \Delta x$$

$$= \sum_{i=1}^{4} \left(0.25(1+i-1)^2 + 1\right)$$

$$= \sum_{i=1}^{4} \left(0.25i^2 + 1\right)$$
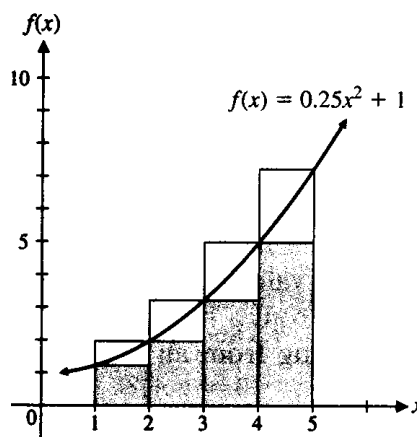

$f(x)$

$f(x) = 0.25x^2 + 1$

On your calculator home screen, press **2nd** followed by **STAT** and then with the right arrow, move to **MATH** followed by pressing **5**. Next. Press **2nd** followed by **STAT** and then with the right arrow, move to **OPS** followed by pressing **5**. Now type in $0.25x^2 + 1, x, 1, 4)$ to complete the sum of the sequence $\text{sum}(\text{seq}(0.25x^2 + 1, x, 1, 4))$.

```
sum(seq(0.25X²+1
,X,1,4))
            11.5
■
```

Repeat the same setup in order to find the sum of the areas of the right rectangles as shown in the figure.

Here $\Delta x = \dfrac{5-1}{4} = 1$ and $x_i = 1 + i(1) = 1 + i$.

So $R_4 = \sum_{i=1}^{4} f(x_i) \cdot \Delta x = \sum_{i=1}^{4} \left(0.25(1+i)^2 + 1\right)$

$f(x)$

$f(x) = 0.25x^2 + 1$

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On your calculator home screen, press **2nd** followed by **STAT** and then with the right arrow, move to **MATH** followed by pressing **5**. Next. Press **2nd** followed by **STAT** and then with the right arrow, move to **OPS** followed by pressing **5**. Now type in $0.25(1 + x^2) + 1, x, 1, 4)$ to complete the sum of the sequence $\text{sum}(\text{seq}(0.25(1 + x^2) + 1, x, 1, 4))$.

```
sum(seq(0.25(1+X
)²+1,X,1,4))
             17.5
∎
```

Note that, in the previous two steps, one can pre-type the functions $Y1 = 0.25x^2 + 1$ and $Y2 = 0.25(1 + x)^2 + 1$ to have the following same results:
To do this on the home screen, press **2nd** followed by **STAT** and then with the right arrow, move to **MATH** followed by pressing **5**. Next. Press **2nd** followed by **STAT** and then with the right arrow, move to **OPS** followed by pressing **5**. Now press **VARS**, followed **Y-VARS** then **1**, and **1** again. Now finish up $\text{sum}(\text{seq}(Y1, x, 1, 4))$.

```
Plot1 Plot2 Plot3
\Y1∎0.25X²+1
\Y2∎0.25(1+X)²+1
∎
\Y3=
\Y4=
\Y5=
\Y6=
```

```
sum(seq(Y1,X,1,4
))
           11.5
∎
```

Similarly press **VARS**, followed **Y-VARS** then **1**, and then **2** this time. Now finish up $\text{sum}(\text{seq}(0.25x^2 + 1, x, 1, 4))$.

```
sum(seq(Y2,X,1,4
))
           17.5
∎
```

Finally, we all know the actual area by using the fundamental theorem of calculus.

There are two ways for finding the actual area.
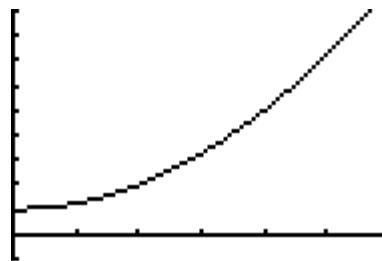
Method 1: Press the **Y=** tab.
After \Y1=, type $0.25x^2 + 1$.

```
Plot1 Plot2 Plot3
\Y1∎0.25X²+1
\Y2=∎
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

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Fix the **WINDOW** as
follows, then press **GRAPH** .

```
WINDOW
 Xmin=0
 Xmax=6
 Xscl=1
 Ymin=-1
 Ymax=9
 Yscl=1
 Xres=1█
```

To calculate the finite integral $\int_1^5 \left(0.25x^2 + 1\right) dx$, Press
2ND followed by **TRACE**, and then **7**. Type in **1** for the
lower limit followed by **ENTER**, and then type in **5** for the
upper limit followed by **ENTER**. The calculator will shade
the area, and hence, $\int_1^5 \left(0.25x^2 + 1\right) dx = \frac{43}{3} \approx 14.33$.

∫f(x)dx=14.333333

Method 2: Press **2ND**
followed by **MODE** to go
back to the home screen.
Now, press **MATH**
followed by **9** and then
complete the following
$\text{fnInt}(0.25x^2 + 1, x, 1, 5)$
followed by **ENTER**. You
can also type
$\text{fnInt}(Y1, x, 1, 5)$.

```
fnInt(0.25X²+1,X
,1,5)
        14.33333333
█
```

```
fnInt(Y₁,X,1,5)
        14.33333333
█
```

Hence, $\int_1^5 \left(0.25x^2 + 1\right) dx = \left(\frac{0.25}{3}x^3 + x\right)\Big|_1^5 = \frac{43}{3} \approx 14.33$.

4. Applications of differentiation and integration

Example 5.1 For $f(x) = -4 - 3x - x^3$, use your calculator to find the slope of the tangent line at
the point $(-1, -6)$.

Press the **Y=** tab. Type
$-4 - 3x^2 - x^3$ on the first line after
\Y1=.

```
Plot1  Plot2  Plot3
\Y₁🭾-4-3X²-X^3
\Y₂=█
\Y₃=
\Y₄=
\Y₅=
\Y₆=
\Y₇=
```
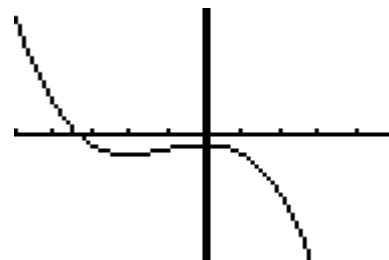
```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-50
 Ymax=50
 Yscl=1
 Xres=1■
```

Next, let us setup the **WINDOW** so we may see a nice **GRAPH**.

We want to find the slope of
$f(x) = -4 - 3x^2 - x^3$ at $x = -1$.

To do this, Press **2ND** followed by **TRACE**, and then **6**. Type -1, and then press **ENTER**.

```
Y1=-4-3X2-X^3




X=-1■
```
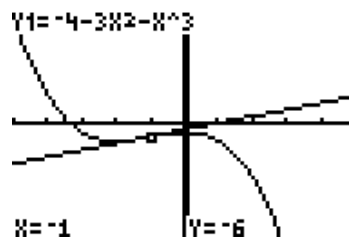
```




dy/dx=2.999999
```
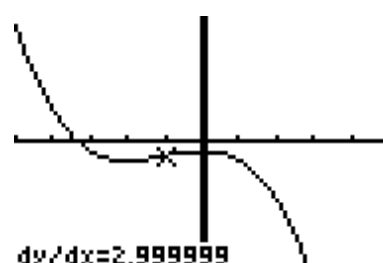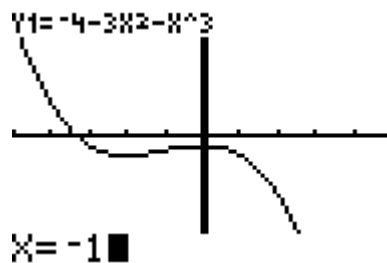
Hence, the slope of the tangent line at the point $(-1, -6)$ is 3, and an equation of the tangent at $(-1, -6)$ is $y + 6 = 3(x + 1)$ or $y = 3x - 3$

Type $3x - 3$ after \Y2=, then press **GRAPH**. Press **TRACE** and then type -1 followed by **GRAPH** to see the (flashing) point of tangency.

```
Plot1 Plot2 Plot3
\Y1 ■ -4-3X2-X^3
\Y2 ■ 3X-3
\Y3 =■
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

```
Y1=-4-3X2-X^3




X=-1    Y=-6
```

Example 5.2 Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

We need to solve the following system of linear equations.

We need to solve the following system of linear equations.

$$\begin{aligned} f(-2) &= 6 \\ f'(-2) &= 0 \\ f(2) &= 0 \\ f'(2) &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} -8a + 4b - 2c + d &= 6 \\ 8a + 4b + 2c + d &= 0 \\ 12a - 4b + c &= 0 \\ 12a + 4b + c &= 0 \end{aligned}$$

We use matrix inverse to find the coefficients a, b, c, and d.

Press the **2nd** tab followed by x^{-1}. With the right arrow, move over to **EDIT** and then press **1**. Enter the dimensions of matrix A as 4 x 4 and then the coefficients of A one number at a time and each number. To move from one number to the next, you can either press **ENTER** or use the calculator arrows. Use the right arrow to see the fourth column or press **2nd** followed by **MODE**, and then **2ⁿᵈ** followed by x^{-1}, then **ENTER**.

```
NAMES MATH EDIT
1■[A]  1×1
2:[B]
3:[C]
4:[D]
5:[E]
6:[F]
7↓[G]
```

```
MATRIX[A] 4 ×4
[ -8    4    -2    ─
[ 8     4    2     ─
[ 12    -4   1     ─
[ 12    4   ██     ─

4,3=1
```

```
MATRIX[A] 4 ×4
-4    -2    1     ]
-4    2     1     ]
-4    1     0     ]
-4    1    ██     ]

4,4=0
```

```
[A]
  [[-8  4  -2  1]
   [8   4   2  1]
   [12  -4  1  0]
   [12  4   1  0]]
■
```

Repeat the above steps to enter matrix B. Press the **2nd** tab followed by x^{-1}. With the right arrow, move over to **EDIT** and then press **2**. Enter the dimensions of matrix B as 4 x 1 and then the coefficients of B one number at a time and each number. To move from one number to the next, you can either press **ENTER** or use the calculator arrows.

```
NAMES MATH EDIT
1:[A]  4×4
2■[B]  1×1
3:[C]
4:[D]
5:[E]
6:[F]
7↓[G]
```

```
MATRIX[B] 4 ×1
[ 6         ]
[ 0         ]
[ 0         ]
[ 0██       ]

4,1=0
```

To see matrix B, press **2nd** followed by **MODE**, and then **2ⁿᵈ** followed by x^{-1}, then **2** followed by **ENTER**.

```
[B]
     [[6]
      [0]
      [0]
      [0]]
■
```

Now matrices A and B are ready to use. The above system of linear equation has a unique solution if and only if the inverse of the coefficient matrix A exists. If not, this methods will fail to provide infinite or extraneous solutions.

In order to solve for the coefficients a, b, c, and d, Press the **2nd** tab followed by x^{-1} and then **ENTER** followed by Press x^{-1}. Again, press **2nd** followed by x^{-1} and then **2** and then press **ENTER**.

```
[A]⁻¹[B]▶Frac
      [[3/16]
       [0   ]
       [-9/4]
       [3   ]]
■
```

Hence, the cubic function is $y = \dfrac{3}{16}x^3 - \dfrac{9}{4}x + 3$.

Example 5.3 Use Newton's method to approximate the roots of the equation $x^4 + x - 4 = 0$.
First, let us graph $f(x) = x^4 + x - 4$ to see how many real roots we need to approximate.

Type $x^4 + x - 4$ on the first line
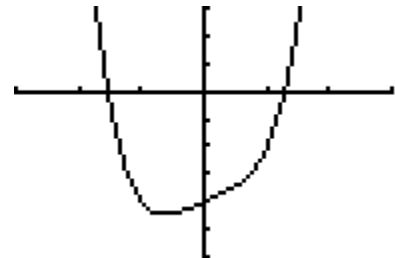after \Y1=.

```
Plot1 Plot2 Plot3
\Y1◻X^4+X-4
\Y2◻nDeriv(Y1,X,
X)■
\Y3=
\Y4=
\Y5=
\Y6=
```

Your window and your graph are as follows:

```
WINDOW
 Xmin=-3
 Xmax=3
 Xscl=1
 Ymin=-6
 Ymax=3
 Yscl=1
 Xres=1■
```

Make sure you do not graph
Y2 (the derivative of Y1) by
covering the = sign and then
pressing **ENTER**.

Hence, the graph shows that there are <u>two</u> real roots. On your home screen, type the following:
Type -1 followed by the **STO→** key (The key above the **ON** key), and then type **X,T,θ,n**,
and hit **ENTER**.

Next press the **STO→** tab followed
by **X,T,θ,n**, then **ALPHA**
followed by **:**. Type $X - Y1/Y2$,
and then press **ENTER**. Keep
pressing **ENTER** until you reach
your desired accuracy, or until the
decimal numbers become identical.

```
-1→X
                      -1
Ans→X:X-Y₁/Y₂
       -2.333331556
       -1.865426506
       -1.615332365
       -1.540102773
■
          -1.865426506
          -1.615332365
          -1.540102773
          -1.533793235
          -1.533751171
          -1.533751169
          -1.533751169
■
```

So, the first zero is approximately -1.533751169.

To find the second zero, go back to your home screen, type the following:

1 followed by the **STO→** key (The key above the **ON** key), and then type **X,T,θ,n**, and hit
ENTER.

Next press the **STO →** tab followed by **X,T,θ,n**, then **ALPHA** followed by **:**. Type $X - Y1/Y2$, and then press **ENTER**. Keep pressing **ENTER** until you reach your desired accuracy, or until the decimal numbers become identical.

```
1→X
              1
Ans→X:X-Y₁/Y₂
     1.39999968
     1.296325969
     1.28394396
     1.283781693
■
```

```
Ans→X:X-Y₁/Y₂
     1.39999968
     1.296325969
     1.28394396
     1.283781693
     1.283781666
     1.283781666
■
```

So, the first zero is approximately 1.283781666.

Notes:

2ND followed by right arrow will take you to the end of the sentence.

2ND followed by left arrow will take you to the beginning of the sentence.

2ND followed by **ENTER** will take you to the last entry.

2ND followed by **(–)** will display the last answer.

 In setting up the **WINDOW** , **xres=1** will calculate the y-value of every pixel, **xres=2** will calculate the y-value of every other pixel, and so on.

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